K–theory for $\mathrm{Sp}(n,1)$

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Let G be a locally compact group. The reduced C^*–algebra $C^*_r(G)$ of G is the norm closure of the image of $L^1(G)$ in $\mathcal{L}(L^2(G))$ acting by left convolution on $L^2(G)$. The full C^*–algebra $C^*(G)$ of G is the enveloping C^*–algebra of the involutive algebra $L^1(G)$. In particular there is a canonical map

$$\lambda\colon C^*(G) \longrightarrow C^*_r(G)\,.$$

The Baum–Connes conjecture [5] predicts the K-theory of $C^*_r(G)$. More precisely there are a geometric group $RK^G_*(\underline{E}G)$, namely the equivariant K–homology with compact supports of the classifying space of proper actions $\underline{E}G$ of G, and assembly maps μ and μ_r such that the following diagram is commutative

$$\begin{array}{ccc} & & K_*(C^*(G)) \\ & \overset{\mu}{\nearrow} & \big\downarrow \lambda_* \\ RK^G_*(\underline{E}G) & \underset{\mu_r}{\longrightarrow} & K_*(C^*_r(G))\,. \end{array}$$

The conjecture asserts that μ_r is an isomorphism. This conjecture is now proved in many cases. However the map λ_* may not be an isomorphism and this is but one of the many difficulties to understand the difference between the two K–theory groups. This arises for example as soon as the non compact group G has Kazhdan property (T). In this case the element of $K_0(C^*(G))$ given by the trivial representation belongs to the kernel of λ_*. This is one reason to look for a better understanding of the K–theory of the full C^*-algebra especially in the presence of property (T). The first computations were given by F. Pierrot [15, 16] who studied the case of the complex Lie groups $\mathrm{SL}_n(\mathbb{C})$ for $n = 3, 4, 5$. He proved in particular that the kernel of λ_* is isomorphic to $\mathbb{Z}$ for $n = 3$ and is an infinitely generated free $\mathbb{Z}$–module when $n = 4, 5$.

Let G be a connected linear semisimple Lie group. The assembly map μ is then easy to construct and the conjecture in this context is also known as the Connes-Kasparov conjecture for connected Lie group which has been checked by A. Wassermann [21]. This conjecture pre-existed the Baum-Connes conjecture for arbitrary locally compact groups, as in the origin the role of the classifying space for proper actions was not fully recognised. For simplicity let us suppose that a maximal compact subgroup K of G is simply connected. Then the isometric

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action of K on the tangent space $\mathfrak{s}$ at the origin eK of G/K lifts to the spin group $\mathrm{Spin}\, d$, ($d = \dim G/K$). Let S be a fundamental $\mathrm{Cliff}_{\mathbb{C}}(\mathfrak{s})$–module and $\gamma\colon \mathfrak{s} \to \mathrm{Cliff}_{\mathbb{C}}(\mathfrak{s}) \simeq \mathrm{End}_{\mathbb{C}} S$ be the canonical map. If d is even we then choose a decomposition $S = S^+ \oplus S^-$ as $\mathrm{Spin}\, d$–module. For any auxiliary finite dimensional representation V of K we form the bundle $\mathcal{S} \otimes \mathcal{V}$ on G/K associated to the representation $S \otimes V$ of K. Let $(X_i)_{i=1,\dots,d}$ be an orthonormal basis of $\mathfrak{s}$. Let us see the X_i's as right invariant vector fields on G/K . The (twisted) Dirac operator D_V is the operator defined, for $f \otimes s \otimes v \in \big(C_c^\infty(G) \otimes S \otimes V\big)^K \simeq \Gamma_c(\mathcal{S} \otimes \mathcal{V})$, by the formula

$$D_V(f \otimes s \otimes v) = \sum_{i=1}^{d} X_i f \otimes \gamma(X_i) s \otimes v \,.$$

Then D_V is a G–invariant first order differential operator on the G–proper space G/K. So by [13] it has an index $\mathrm{ind}_a D_V \in K_d(C^*(G)$ that only depends on the K–homolgy class $[D_V] \in K_d^G(G/K) = R(K)$ of D_V. Extending this by additivity we get the desired morphism

$$\begin{array}{rcl} \mu\colon R(K) & \to & K_d(C^*(G)) \\ V & \mapsto & \mathrm{ind}_a D_V \,. \end{array}$$

In this paper we compute the K-theory of $C^*(G)$ for the group $G = \mathrm{Sp}(n,1)$, $n \geq 2$. The same arguments are also valid for the exceptional group $F_{4(-20)}$. These groups are among the simplest examples of simple Lie groups for which λ_* is not an isomorphism. In fact for the others simple real rank one Lie groups the map λ actually induces a KK-equivalence [11]. Our method is based on the full knowlegde of the unitary dual of G. This work has been done by Baldoni Silva [3]. It appears that the repesentations that are not in the reduced dual are the complementary series and the so called « isolated » series. These are Langlands quotients that are not at the end of the complementary series of the principal series in which they appear. The trivial representation is one of these « isolated » series. It is the only one when $n = 2$ and there are countably many others which are infinite dimensional when $n \geq 3$.

Theorem. (see Theorem 1.14) *Let $G = \mathrm{Sp}(n,1)$ and $\mathcal{I}$ be the set of « isolated » series. The morphism $p = \lambda \oplus (\oplus_{\mathcal{I}} \pi)$*

$$C^*(G) \xrightarrow{\;p\;} C_r^*(G) \oplus \left(\underset{\pi \in \mathcal{I}}{\oplus} \mathcal{K}(\mathcal{H}_\pi) \right) \tag{1}$$

induces an isomorphism in K–theory.

In particular the kernel of λ_* is a free $\mathbb{Z}$–module with a set of generators in bijective correspondance with the « isolated » series.

In the second part of this paper we describe explicitely the graph of the non zero map

$$K_0(C_r^*(G)) \xrightarrow{(\oplus_{\mathcal{I}} \pi_*) \circ \mu \circ \mu_r^{-1}} \oplus_{\pi \in \mathcal{I}} \mathbb{Z} \,.$$

It appears that for a given irreducible representation V of K there may be one or more « isolated » series π such that $\pi_*\big(\mathrm{ind}_a D_V\big) \neq 0$. This phenomena can be understood as the fact that the « isolated » series may appear in some Dolbeault cohomology spaces over the proper G–space G/T where T is a compact Cartan subgroup of G. The reader is referred to the introduction of [18] for more details on this subject.

The results are essentially taken from my thesis [17]. It is a pleasure to thank my thesis advisor P. Julg for introducing me to the subject of K–theory of group C^*–algebras. I also would like to thank G. Skandalis who indicated to me the KK–improvement in the statement of Theorem 1.14.

1 K–theory for $\mathrm{Sp}(n,1)$

1.1 Cartan subgroups and root systems

Let $n \geq 2$. The Lie group $G = \mathrm{Sp}(n,1)$ is the group of linear transformations of the right $\mathbb{H}$–vector space $\mathbb{H}^{n+1}$ preserving the sesquilinear form

$$(u,v) = \sum_{i=1}^{n} \overline{u}_i v_i - \overline{u}_{n+1} v_{n+1}\,, \qquad u = (u_i), v = (v_i) \in \mathbb{H}^{n+1}\,.$$

In other word, if we identify $g \in G$ with its matrix in the canonical basis, then

$$G = \mathrm{Sp}(n,1) = \{g \in M_{n+1}(\mathbb{H})\,;\, g^* J g = J\}$$

where g^* is the conjugate transposed matrix of g and $J = \left(\begin{smallmatrix} 1_n & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Let $\mathfrak{g}$ be the Lie algebra of G. In the same way Lie groups will be denoted by upcase latin letters and their Lie algebras with the same letters using german script. The fixed point set of the Cartan involution θ defined by $\theta(g) = JgJ$ is a maximal compact subgroup K of G isomorphic to Sp(n) $\times$ Sp(1). The corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ is given by

$$\begin{aligned} \mathfrak{k} &= \{\left(\begin{smallmatrix} M & 0 \\ 0 & q \end{smallmatrix}\right)\,;\, M + M^* = 0\,,\, q + \overline{q} = 0\}\,, \\ \mathfrak{s} &= \{\left(\begin{smallmatrix} 0 & X \\ X^* & 0 \end{smallmatrix}\right)\,;\, X \in \mathbb{H}^n\}\,. \end{aligned}$$

The abelian Lie algebra $\mathfrak{t} = \mathfrak{so}(2) \times \cdots \times \mathfrak{so}(2) \subset \mathfrak{k}$ is a Cartan subalgebra in both $\mathfrak{k}$ and $\mathfrak{g}$. In particular G has a compact Cartan subgroup. We denote by Δ the root system of $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$, and $\Delta_c \subset \Delta$ the root system of $(\mathfrak{k}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. The roots in Δ_c are called compact roots, and $\Delta_n = \Delta \setminus \Delta_c$ is the set of noncompact roots. If we are given Δ^+ a positive root system of Δ then we denote by ρ (or $\rho(\Delta^+)$ if necessary) half the sum of these positive roots. We will also use the notations ρ_c and ρ_n for half the sum of compact or noncompact positive roots respectively. The group G has an Iwasawa decomposition in the following way. Let $\mathfrak{a}$ be maximal abelian subspace of $\mathfrak{s}$ and M the centralizer of $\mathfrak{a}$ in K. For example we may take $\mathfrak{a}$ to be the set of matrices

$$\mathfrak{a} = \left\{ H_t = \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ t & 0 & 0 \end{pmatrix}, t \in \mathbb{R} \right\} \subset \mathfrak{s}$$

then we find that M is the group of matrices of the form

$$M = \left\{ g = \begin{pmatrix} q & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & q \end{pmatrix}, \, m \in \mathrm{Sp}(n-1)\,,\, |q| = 1 \right\} \subset K\,.$$

Let $\mathfrak{b} \subset \mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{m}$ and $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{a}$. Let Φ be the root system of $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ and $\Phi_\mathfrak{m}$ the root system of $(\mathfrak{m}_\mathbb{C}, \mathfrak{b}_\mathbb{C})$. If Φ^+ is a positive system for Φ we denote by $\Phi^+_\mathfrak{m} = \Phi^+ \cap \Phi_\mathfrak{m}$ the corresponding system of positive roots for $\Phi_\mathfrak{m}$. The nilpotent algebra $\mathfrak{n}$ is the sum of the root spaces of the system $\Phi_\mathfrak{a}$ containing the non-zero restrictions to $\mathfrak{a}$ of positive roots of Φ. Then an Iwasawa decomposition is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Note that $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is the Lie algebra of a minimal parabolic subgroup P of G. For $i = 1, \ldots, n+1$ let ε_i be the linear form on $\mathfrak{t}_\mathbb{C}$ given by $\varepsilon_i(\mathrm{diag}(t_1, \ldots, t_{n+1})) = t_i$. Then

$$\begin{aligned} \Delta &= \{\pm\varepsilon_i \pm \varepsilon_j\,;\, 1 \leq i < j \leq n+1\} \cup \{\pm\varepsilon_i\,;\, 1 \leq i \leq n+1\} \\ \Phi &= \{\pm e_i \pm e_j\,;\, 1 \leq i < j \leq n+1\} \cup \{\pm e_i\,;\, 1 \leq i \leq n+1\} \end{aligned}$$

where $e_i = \varepsilon_{i-1} \circ \mathrm{Ad} u^{-1}$ for $i = 3, \ldots, n+1$ and $e_1 = \varepsilon_1 \circ \mathrm{Ad} u^{-1}$ and $e_2 = -\varepsilon_{n+1} \circ \mathrm{Ad} u^{-1}$ with $u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & I & 0 \\ -1 & 0 & 1 \end{pmatrix} \in G_{\mathbb{C}}$. We have $\mathfrak{a}^* = \mathbb{R}(e_1 + e_2)$ and we may thus identify $\mathfrak{a}^*_{\mathbb{C}}$ with $\mathbb{C}$. We also note that $(e_1 + e_2)(H_t) = -2t$. We choose the following positive root system :

$$\begin{array}{rcl} \Delta^+ & = & \{\varepsilon_i \pm \varepsilon_j \,;\, 1 \leq i < j \leq n+1\} \cup \{\varepsilon_i \,;\, 1 \leq i \leq n+1\} \text{ and} \\ \Phi^+ & = & \{e_i \pm e_j \,;\, 1 \leq i < j \leq n+1\} \cup \{e_i \,;\, 1 \leq i \leq n+1\} . \end{array} \tag{2}$$

1.2 Admissible and unitary representations

We now describe irreducible admissible and irreducible unitary representations of G.

The set $\hat{H}$ of equivalence classes of irreducible representations of a compact connected Lie group H is parametrized by their highest weight relative to a given positive root system. We will freely identify these representations with their highest weight. The highest weight ξ of an irreducible representation of M relative to $\Phi^+_{\mathfrak{m}}$ is given by $\xi = b(e_1 - e_2) + \sum_{i=3}^{n+1} b_i e_i$ with $2b, b_i \in \mathbb{N}$ and $b_3 \geq \cdots \geq b_{n+1}$. The highest weight μ of an irreducible representation of K relative to Δ^+_c is given by $\mu = \sum_{i=1}^{n+1} \mu_i \varepsilon_i$ with $\mu_i \in \mathbb{N}$ and $\mu_1 \geq \cdots \geq \mu_n$.

Let π be a continuous representation of G in a Hilbert space whose restriction to K is unitary. Then $\pi|_K = \sum_{\eta \in \hat{K}} n_\eta \eta$. The representation π is said to be admissible if for all $\eta \in \hat{K}$ we have $n_\eta < \infty$.

Let $\xi \in \hat{M}$ and $\nu \in \mathfrak{a}^*_{\mathbb{C}}$. We construct the induced representation from P to G, $\pi_{\xi,\nu} = \mathrm{Ind}_P^G \xi \otimes e^\nu \otimes 1$. Let $\delta_{\mathfrak{a}}$ be half the sum of positive roots in $\Phi_{\mathfrak{a}}$. A dense subspace of the representation space of $\pi_{\xi,\nu}$ is

$$\left\{ F \colon G \to V_\xi \text{ continuous} \,;\, F(gman) = e^{-(\nu+\rho_{\mathfrak{a}}) \log a} \xi(m)^{-1} F(g) \right\},$$

with the norm $\|F\| = \int_K |F(k)|^2 dk$ and G acts on the left. These representations are the principal series. They are normalized to be unitary if ν is imaginary. Two principal series $\pi_{\xi,\nu}$ and $\pi_{\xi',\nu'}$ with $\nu' \neq \nu$ are equivalent if and only if $\xi' = \xi$ and $\nu' = -\overline{\nu}$. For any ν such that $\mathrm{Re}\nu > 0$ the representation $\pi_{\xi,\nu}$ has a unique irreducible quotient $J_{\xi,\nu}$. It is called a Langlands quotient.

Proposition 1.1. *[2] If $\mathrm{Re}\nu = 0$ and $\mathrm{Im}\nu > 0$ then $\pi_{\xi,\nu}$ is irreducible. The representation $\pi_{\xi,0}$ is reducible if and only if $\xi = b(e_1 + e_2) + \sum b_i e_i$ is such that $b \in \mathbb{N} + 1/2$ and $b + 1/2 \neq b_j + n - j + 2$ for $j = 3, \ldots, n+1$. In such a case the representation $\pi_{\xi,0}$ is the direct sum of two irreducible unitary representations called limits of discrete series. Let $\hat{M}_{red}$ the set of $\xi \in \hat{M}$ such that $\pi_{\xi,0}$ is reducible.*

The Langlands classification theorem for irreducible admissible representations of G reads as follows.

Theorem 1.2. *[14, Theorem 14.92] Let $G = \mathrm{Sp}(n, 1)$. Up to equivalence the irreducible admissible representations of G are :*

1. *the discretes series i.e. irreducible representations whose coefficients are square integrable,*

2. *the limits of discrete series,*

3. *the unitary principal series i.e. the representation* $\pi_{\xi,\nu}$ *with* $\mathrm{Re}\nu = 0$ *and* $\mathrm{Im}\nu > 0$ *or* $\nu = 0$ *if* $\xi \in \hat{M}_{red}$*,*

4. *the Langlands quotients.*

The first three series are unitary representations and constitute the admissible tempered dual of G. It is the dual of the reduced C^*–algebra of G.

Irreducible unitary representations of semi–simple Lie groups are admissible by [9, Theorem 15.5.6]. To classify unitary representations of G it then remains to determine the unitarizable Langlands quotients. Baldoni Silva has obtained the following theorem.

Theorem 1.3. *[3, Theorem 7.1] Let* $\xi = b(e_1 + e_2) + \sum b_j e_j \in \hat{M}$ *and* ν *such that* $\mathrm{Re}\,\nu > 0$. *If* $J_{\xi,\nu}$ *is unitarizable then* $\mathrm{Im}\,\nu = 0$.

1. *If* $\xi \in \hat{M}_{\mathrm{red}}$ *then for any* $\nu > 0$, $J_{\xi,\nu}$ *is not unitarizable. In this case we set* $\nu(\xi) = 0$.

2. *If* $\xi \notin \hat{M}_{\mathrm{red}}$ *then there exists* $\nu(\xi) > 0$ *so that if* $0 < \nu \leq \nu(\xi)$ *then* $J_{\xi,\nu}$ *is unitarizable. Moreover if* $\nu < \nu(\xi)$ *then* $\pi_{\xi,\nu}$ *is irreducible. These series are called the complementary series. The Langlands quotients* $J_{\xi,\nu(\xi)}$ *are said to be at the end of the complementary series.*

3. *If* $\xi \hat{M}_{\mathrm{red}}$ *and is such that* $b = 0$ *and* $b_{n+1} = 0$ *then* $J_{\xi,\nu(\xi)+1}$ *is unitarizable. These representations are called the isolated series.*

4. *Any unitarizable Langlands quotient is as in 2) or 3).*

1.3 Infinitesimal characters

To understand the topology on the dual $\hat{G}$ of G in the next section we will use the notion of infinitesimal character. See Knapp's book [14] for details. Let G be a linear connected semisimple Lie group and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let $U(\mathfrak{h})$ be the enveloping Lie algebra of $\mathfrak{h}_{\mathbb{C}}$ and $\mathfrak{Z}$ the centre of the enveloping Lie algebra of $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be a Cartan decomposition, $(X_1, \ldots, X_d)$ be an orthonormal basis of $\mathfrak{s}$ and $(Y_1, \ldots, Y_m)$ an orthonormal basis of $\mathfrak{k}$ relative to the Killing form. Then

$$\Omega = -\sum_{j=1}^{m} Y_j^2 + \sum_{i=1}^{d} X_i^2$$

is an element of $\mathfrak{Z}$ called the Casimir operator. It does not depend of the choice of the basis. In particular $\mathfrak{Z} \neq 0$. Harish-Chandra constructs an homomorphism $\gamma_{\mathfrak{h}} : \mathfrak{Z} \to U(\mathfrak{h})^W$ where $W = W(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$ is the Weyl group. Given $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ we denote by

$$\chi_\lambda : \mathfrak{Z} \to \mathbb{C}$$

the homomorphism defined by $\chi_\lambda(Z) = \lambda(\gamma_{\mathfrak{h}}(Z))$.

1. Any homomorphism from $\mathfrak{Z}$ to $\mathbb{C}$ is obtained in this way.

2. $\chi_\lambda = \chi_\mu$ if and only if there exists $\sigma \in W$ such that $\lambda = \sigma\mu$.

3. Let $\mathfrak{h}'$ be an other Cartan subalgebra of $\mathfrak{g}$ and $\mathrm{Ad}x : \mathfrak{h}_{\mathbb{C}} \to \mathfrak{h}'_{\mathbb{C}}$ for some $x \in G_{\mathbb{C}}$. We have $\gamma_{\mathfrak{h}'} = \mathrm{Ad}x.\gamma_{\mathfrak{h}}$.

If $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$ can be lifted to a character of $H = \exp \mathfrak{h}$ we say that λ is integral. If $(\lambda, \alpha) \neq 0$ for all roots α of $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{h}_{\mathbb{C}}$ we say that λ is regular, and singular otherwise. We use the same word for χ satisfying $\chi = \chi_\lambda$.

We say that a representation π of G has an infinitesimal character if there exists λ such that

$$\forall Z \in \mathfrak{Z}\,, \quad \pi(Z) = \chi_\lambda(Z)\,.$$

The infinitesimal character of π is denoted by χ_π.

Let $G = \mathrm{Sp}(n, 1)$. The irreducible admissible representations of G and the representations $\pi_{\xi,\nu}$ have an infinitesimal character. Let $\delta_{\mathfrak{m}}$ be the half sum of positive roots of $\Phi_{\mathfrak{m}}$. It follows from Knapp's book [14, Proposition 8.22] that the infinitesimal character of $\pi_{\xi,\nu}$ is given by $\chi_{\Lambda_{\xi,\nu}}$ with $\Lambda_{\xi,\nu} = \xi + \delta_{\mathfrak{m}} + \nu$. The linear form $\Lambda_{\xi,\nu}$ is the Langlands parameter of $\pi_{\xi,\nu}$. It is very easy to check that if we set $\Lambda_{\xi,\nu} = \sum_{i=1}^{n+1} a_i e_i$ with $\xi = b(e_1 - e_2) + \sum b_i e_i$ and $\nu = c(e_1 + e_2)$ then

$$\begin{cases} a_1 = c + b + 1/2\,, \\ a_2 = c - b - 1/2\,, \\ a_i = b_i + n - i + 2\,, \quad 3 \leq i \leq n+1\,. \end{cases} \tag{3}$$

1.4 Topology of the unitary dual

The description we give of the K–theory for the maximal C^*–algebra of $\mathrm{Sp}(n,1)$ is based on the full knowledge of the unitary dual as well as on a basic understanding of its topology. We then study the Fell–Jacobson topology on G.

Let G be a locally compact group. The set $\mathrm{Rep}(G)$ of unitary representations of G is endowed with the Fell topology which may be described as follows. The set $\hat{G}$ inherits the induced topology. Let C be a compact subset of G, $\varepsilon > 0$, $(\pi, \mathcal{H}_\pi) \in \mathrm{Rep}(G)$ and $\xi_1, \dots \xi_n$ an orthonormal family in $\mathcal{H}_\pi$. Let us consider the set of representations $(\sigma, \mathcal{H}_\sigma) \in \mathrm{Rep}(G)$ such that there exists an orthonormal family $\eta_1, \dots, \eta_n$ in $\mathcal{H}_\sigma$ with

$$(\forall i, j = 1, \dots, n) \qquad \underset{g \in C}{Sup}\, |\langle \eta_i ; \sigma(g)\eta_j\rangle - \langle \xi_i ; \pi(g)\xi_j\rangle| < \varepsilon\,. \tag{4}$$

These sets $V(\pi, C, \varepsilon, \xi_i)$ constitute a fundamental basis of neighborhoods of π.

Let K be a compact subgroup of G. For any $\tau \in \hat{G}$ we write

$$\tau_{|K} = \oplus_{\rho \in \hat{K}} V^\tau_\rho$$

where V^τ_ρ is the space on which τ acts via ρ.

Lemma 1.4. *Let $\pi \in \hat{G}$ and $\rho \in \hat{K}$. We assume that for any $\tau \in \hat{G}$ the space V^τ_ρ has finite dimension. Then $\hat{G}_{\rho,\pi} = \{\sigma \in \hat{G} : \dim V^\pi_\rho \leq \dim V^\sigma_\rho\}$ is open in $\hat{G}$.*

Proof. Let $\sigma \in \hat{G}_{\rho,\pi}$ and let us show that $V(\sigma, K, \varepsilon, \eta_i) \subset \hat{G}_{\rho,\pi}$ for some $\varepsilon > 0$ small enough and an appropriate choice of the η_i's. Let $(\xi_i)_{i=1}^n$ be an orthonormal family in V^σ_ρ. Let $\varepsilon > 0$ and $\sigma' \in V(\sigma, K, \varepsilon, \xi_i)$. There exists orthonormal vectors $\eta_1, \dots, \eta_n$ as in equation (4) with $C = K$. For $\tau \in \hat{G}$ we denote by P^τ_ρ the orthogonal projection on V^τ_ρ. Then

$$P^\tau_\rho v = d_\rho \int_K \overline{tr\rho(k)}\tau(k)v\, dk\,.$$

Let $\eta'_i = P_\rho^{\sigma'}\eta_i$. We have to show that the family (η'_i) is free. Let $\{i,j\} \subset \{1,\dots,n\}^2$. Then

$$\begin{array}{rcl} |\langle \eta_i ; \eta'_j\rangle & - & \delta_{i,j}| \\ & = & |\langle \eta_i ; \eta'_j\rangle - \langle \xi_i ; \xi_j\rangle| \\ & = & |\langle \eta_i ; d_\rho \int_K \overline{tr\rho(k)}\sigma'(k)\eta_j\rangle \, dk - \langle \xi_i ; d_\rho \int_K \overline{tr\rho(k)}\sigma(k)\xi_j\rangle \, dk| \\ & \leq & d_\rho \int_K \overline{tr\rho(k)} \, |\langle \eta_i ; \sigma'(k)\eta_j\rangle - \langle \xi_i ; \sigma(k)\xi_j\rangle| \, dk \\ & \leq & \varepsilon \, d_\rho \int_K |\overline{tr\rho(k)}| \, dk \end{array}$$

So if $\varepsilon > 0$ is small enough the family (η'_i) is free. In conclusion $V_\rho^\pi \subset V_\rho^\sigma$. □

Let $G = \mathrm{Sp}(n,1)$. The group G is liminal and the Jacobson topology on $\hat{G}$ coincides with the Fell topology (see [9]). In particular, for any point π in $\hat{G}$ the set $\{\pi\}$ is closed.

Lemma 1.5. *Let (π_n) be a sequence converging to π. Let us suppose that for any n the representation π_n is a subquotient of π_{ξ,ν_n} with ν_n real. Then the sequence (ν_n) converges.*

Proof. Let us recall that Ω is the Casimir operator. It is well known that the fonction $\sigma \mapsto \sigma(\Omega)$ is continuous on $\hat{G}$ (this is a lemma due to Dixmier, see [6] for a proof). But $\pi_n(\Omega) = \pi_{\xi,\nu_n}(\Omega)$. We note $|\,|$ the norm on $(\mathfrak{a} + i\mathfrak{b})^*$ obtained by restricting the Killing form and δ the half sum of the positive roots in Φ^+. For any $\nu > 0$ we have

$$\begin{array}{rcl} \pi_{\xi,\nu}(\Omega) & = & \chi_{\Lambda_{\xi,\nu}}(\Omega) I \\ & = & (|\Lambda_{\xi,\nu}|^2 - |\delta|^2) I \\ & = & (|\xi + \delta_{\mathfrak{m}}|^2 + |\nu|^2 - |\delta|^2) I \, . \end{array}$$

The second equality follows for example from [14, Lemma 12.28]. We conclude that the sequence (ν_n) converges. □

Let π be an admissible representation of G.

Proposition 1.6. *(Harish-Chandra) For any $f \in C_c^\infty(G)$, the operator*

$$\pi(f) = \int_G f(g)\pi(g) d\, g$$

is trace class and the linear form

$$f \mapsto \mathrm{Trace}(\pi(f))$$

defines a distribution on $C_c^\infty(G)$. It is given by a fonction θ_π defined on a dense open subset G' of G on which it is analytic.

As a consequence any $\pi \in \hat{G}$ extends to $\pi \colon C^*(G) \to \mathcal{K}(\mathcal{H}_\pi)$.

Proposition 1.7. *[14, Proposition 10.18] Let $\pi = \pi_{\xi,\nu}$. We have for any $x \in G'$*

$$\theta_\pi(x) = \theta_{\xi,\nu}(x) = \begin{cases} D^{-1}(h)((e^\nu + e^{-\nu}) \otimes \mathrm{ch}\xi|_B)(h) & \text{if } x = ghg^{-1} \text{ for } h \in BA \, , \\ 0 & \text{otherwise,} \end{cases}$$

where the denominator D does not depend on ν nor ξ. In particular $\theta_{\xi,\nu}$ is continuous in ν.

Proposition 1.8. *Let (ν_n) converging to ν. The sequence (π_{ξ,ν_n}) converges to $\sigma \in \hat{G}$ if and only if σ is a subquotient of $\pi_{\xi,\nu}$.*

Proof. By Proposition 1.7 the distribution

$$f \longmapsto \operatorname{Trace} \pi(f)$$

on $C_c^\infty(G)$ is defined for $\pi = \pi_{\xi,\nu}$ by a locally integrable function $\theta_{\xi,\nu}$ whose module is bounded by a constant times $|\theta_{\xi,0}|$ which is locally integrable. The dominated convergence theorem then gives for any $f \in C_c^\infty(G)$

$$\lim_{\nu_n \to \nu} \operatorname{Trace} \pi_{\xi,\nu_n}(f) = \operatorname{Trace} \pi_{\xi,\nu}(f)$$

The conclusion then follows from [10, Corollary 2 of Theorem 2.3 and Lemma 3.4]. These fundamental results of Fell are recalled in the next proposition. □

For $\mu \in \hat{K}$ let $p_\mu : k \mapsto \dim \mu \operatorname{Trace} \mu(k)$ be the associated projector in $C^\infty(K)$. As $C^\infty(K)$ acts by convolution on $C_c^\infty(G)$ we can define an involutive subalgebra of $C_c^\infty(G)$ by

$$\mathfrak{A} = \sum_{\mu_1,\mu_2} p_{\mu_1}.C_c^\infty(G).p_{\mu_2}\,.$$

This subalgebra is dense in $C_c^\infty(G)$.

Proposition 1.9. *1. Let $f \in \mathfrak{A}$. There exists $n \in \mathbb{N}$ such that for any $\pi \in \hat{G}$ we have $\operatorname{Rank}\pi(f) \leq n$.*

2. Let $(\pi_n)_{n\in\mathbb{N}}, \sigma_1, \ldots, \sigma_r$ be elements of $\hat{G}$ and assume that for any $f \in \mathfrak{A}$ we have

$$\lim_{n\to\infty} \operatorname{Trace} \pi_n(f) = \sum_{i=1}^{r} \operatorname{Trace} \sigma_i(f)\,.$$

Then the sequence (π_n) converges to $\sigma \in \hat{G}$ if and only if there exists i such that $\sigma = \sigma_i$.

Remark 1.10. *Together with the fact that the distributions θ_π for π admissible irreducible are linearly independant it results in particular that any subquotient of $\pi_{\xi,\nu(\xi)}$ is unitarizable.*

We are now ready to describe the Fell topology on $\hat{G}$. Let us call

1. $\mathcal{C}$ the set of complementary series,
2. $\mathcal{D}$ the set of discrete series,
3. $\mathcal{E}$ the set of end points of the complementary series,
4. $\mathcal{I}$ the set of « isolated » series,
5. $\mathcal{L}$ the set of limits of discrete series and
6. $\mathcal{P}$ the set of unitary principal series.

Theorem 1.11. *1. The sets $\mathcal{I}$, $\mathcal{D}$, $\mathcal{L}$ and $\mathcal{E}$ are closed and discrete in $\hat{G}$.*

2. Let $\xi \in \hat{M}$. The closure of $\mathcal{PC}(\xi) = \{\pi_{\xi,\nu} \in \mathcal{P} \cup \mathcal{C}\}$ is the union of $\mathcal{PC}(\xi)$ and of the subquotients of $\pi_{\xi,\nu(\xi)}$.

Proof. Let $\pi \in \hat{G}$ and (π_n) converging to π. If μ is some K-type of π then we can assume that μ is a K-type of π_n thanks to lemma 1.4. Any unitary irreducible representation of G appears as a subquotient in a principal series $\pi_{\xi,\nu}$ with $\xi \in \hat{M}$ and $\nu \in \mathfrak{a}^*_{\mathbb{C}}$. Then there exists a (not unique) sequence (ξ_n, ν_n) such that π_n is a subquotient of π_{ξ_n,ν_n} and this representation also contains μ as a K-type. By Frobenius reciprocity this implies that $\xi_n \subset \mu|_M$. In particular there is only a finite number of ξ_n satisfying the above condition and the sequence (ξ_n) is a finite union of constant subsequences. It suffices then to look at these subsequences separately. So we assume that $\xi_n = \xi$ is constant.

By Lemma 1.5 the sequence ν_n converges. Let $\pi \in \mathcal{I}$ (respectively $\mathcal{D}$, $\mathcal{L}$, $\mathcal{E}$). The infinitesimal character of π is integral. So if n is big enough and because (ν_n) converges we can assume that either π_n has the same infinitesimal character than π or that this is not integral. In the first case $\pi_n = \pi$ because points are closed and there exists only a finite number of admissible representations with a given infinitesimal character. In the second case $\pi_n = \pi_{\xi,\nu_n}$ because the principal series whose infinitesimal character are not integral are irreducible. □

Remark 1.12. *This theorem implies in particular that for any $f \in C_c^\infty(G)$ the map $\pi \to \|\pi(f)\|$ tends to 0 as $\pi \in \mathcal{I}$ tends to infinity in the discrete set of « isolated » series.*

For a complete study of the Fell topology it remains to describe explicitely the subquotients of the principal series $\pi_{\xi,\nu}$. This work has been done M.W. Baldoni–Silva [2]. Using an explicit knowledge of the length of the complementary series one can prove [17] that the subquotients of $\pi_{\xi,\nu(\xi)}$ could be either elements of $\mathcal{E}$, $\mathcal{D}$, $\mathcal{L}$, or even $\mathcal{C}$ or $\mathcal{I}$. This means in particular that the topology on $\mathcal{P} \cup \mathcal{C}$ is not the topology induced from the parameter space $\hat{M} \times \mathfrak{a}^*_{\mathbb{C}}$ and also that if $\pi \in \mathcal{I}$ then the set $\{\pi\}$ may be non-open. However we do not need these results to determine the K–theory of $C^*(G)$. We will recall in the next section the part of these results we need to determine the range of the full Baum-Connes map. In this section we only need the following result of semisimple theory.

Proposition 1.13. *[14, Proposition 8.21] If $\pi = J_{\xi',\nu'}$ is a subquotient of $\pi_{\xi,\nu}$ with $\xi' \neq \xi$ then $\nu > \nu'$.*

1.5 Main result

Theorem 1.14. *The morphism $p = \lambda \oplus (\oplus_{\pi \in \mathcal{I}} \pi)$*

$$C^*(G) \xrightarrow{p} C^*_r(G) \oplus \left(\bigoplus_{\pi \in \mathcal{I}} \mathcal{K}(\mathcal{H}_\pi) \right) \tag{5}$$

induces a KK–equivalence.

Let us first recall [9] that the set of closed ideals in a C^*–algebra A is in one to one correspondance with the set of open subsets of the spectrum $\hat{A}$ of this C^*–algebra. This correspondance associates to a closed ideal I of A the set of irreducible representations π of A such that $\pi|_I \neq 0$. In the same way the quotients of A by closed ideals correspond to the closed subsets in the spectrum of A. The open set corresponding to $\operatorname{Ker} p$ is

$$(\operatorname{Ker} p)\hat{} = \mathcal{C} \cup \mathcal{E}\,.$$

Remark that the map p is well defined thanks to remark 1.12.

We first show that the map p_* induces an isomorphism in K–theory. Writing the six term exact sequence in K–theory associated to the short exact sequence

$$0 \to \operatorname{Ker} p \to C^*(G) \xrightarrow{p} C^*_r(G) \oplus \left(\bigoplus_{\pi \in \mathcal{I}} \mathcal{K}(\mathcal{H}_\pi) \right) \to 0\,,$$

we see that it suffices to show that $K_*(\operatorname{Ker} p) = 0$.

Lemma 1.15. *The set*

$$\{\nu > 0; \exists \xi \in \hat{M},\, \pi_{\xi,\nu} \text{ reducible and } J_{\xi,\nu} \in \mathcal{C} \cup \mathcal{E}\}$$

is finite.

Proof. If $\pi_{\xi,\nu}$ is reducible then $\Lambda_{\xi,\nu} = \sum a_i e_i$ is integral. But this can arise if and only if $a_i \in \mathbb{Z}$. On the one hand the equations (3) read here

$$\begin{cases} a_1 = \nu + b + 1/2 \\ a_2 = \nu - b - 1/2 \end{cases}$$

This implies $\nu \in 1/2\mathbb{N}$. On the other hand if $J_{\xi,\nu}$ is unitary $\nu \le \delta_{\mathfrak{a}} = n + 1/2$ where $\delta_{\mathfrak{a}}$ is the half sum of positive roots of $\Phi_{\mathfrak{a}}$ thanks to [3, Theorem 1.2]. □

Definition 1.16. *let $\nu_1, \dots, \nu_k$ be the elements of the finite set*

$$\{\nu > 0; \exists \xi \in \hat{M},\, \pi_{\xi,\nu} \text{ reducible and } J_{\xi,\nu} \in \mathcal{C} \cup \mathcal{E}\}\,,$$

ordered such that

$$\nu_1 < \cdots < \nu_k\,.$$

Let us set $\nu_0 = 0$.

Definition 1.17. *For any $l = 0, \dots, k$ let*

$$\hat{G}_l = \{\pi = J_{\xi,\nu} \in \mathcal{C} \cup \mathcal{E},\, \nu > \nu_l\}\,.$$

We have $\hat{G}_0 = \mathcal{C} \cup \mathcal{E}$ and $\hat{G}_k = \varnothing$.

Proposition 1.18.

1. $(\operatorname{Ker} p)\hat{} = \hat{G}_0 \supset \cdots \supset \hat{G}_k = \varnothing$.
2. *$\hat{G}_l$ is open in $\hat{G}_0$.*
3. *For any $\xi \in \hat{M}$ and $l < k$ let*

$$\hat{G}_l(\xi) = \{\pi = J_{\xi,\nu} \in \mathcal{CB}(\xi)\,;\, \nu_l < \nu \le \nu_{l+1}\}\,,$$

with $\mathcal{CB}(\xi) = \{J_{\xi,\nu}\,;\, 0 < \nu \le \nu(\xi)\}$. Then $\hat{G}_l \setminus \hat{G}_{l+1} = \sqcup \hat{G}_l(\xi)$ et $\hat{G}_l(\xi)$ is open and closed in $\hat{G}_l \setminus \hat{G}_{l+1}$.

Proof. The first assertion is trivial. The second assertion and the beginning of the third assertion follow immediately from proposition 1.13. It remains to show that $\hat{G}_l(\xi)$ is closed in $\hat{G}_l \setminus \hat{G}_{l+1}$. Let (π_n) be a sequence converging to $\pi = J_{\xi,\nu} \in \hat{G}_l(\xi)$. The same argument as in the proof of theorem1.11 shows that we may assume that $\pi_n = J_{\xi',\nu_n}$ and ν_n converges to ν if $\xi' = \xi$ and to $\nu(\xi')$ otherwise. Because of Proposition 1.13 we have $\nu(\xi') \ge \nu_{l+1}$. So if $\pi_n \in \hat{G}_l \setminus \hat{G}_{l+1}$ we obtain $\pi_n = \pi_{\xi,\nu_n} \in \hat{G}_l(\xi)$. □

Let J_l be the closed ideal of $\operatorname{Ker} p$ associated to the open set $\hat{G}_l$. We obtain a decreasing sequence of closed ideals in $\operatorname{Ker} p$

$$\operatorname{Ker} p = J_0 \supset \cdots \supset J_k = 0\,.$$

Now we remark that $K_*(J_k) = 0$. So if we prove that for any $l = 0, \ldots, k-1$, $K_*(J_l/J_{l+1}) = 0$ then we will be able to conclude - using again the six term exact sequence - that the inclusion $J_{l+1} \subset J_l$ induces an isomorphism in K–theory. It will follow that

$$K_*(\operatorname{Ker} p) = K_*(J_0) = \cdots = K_*(J_k) = 0\,.$$

For any $\xi \in \hat{M}$ let $A_{l,\xi}$ be the ideal of J_l/J_{l+1} associated to $\hat{G}_l(\xi)$. The preceding proposition implies that $J_l/J_{l+1} = \oplus A_{l,\xi}$. It then suffices to show that $A_{l,\xi}$ is zero in K–theory.

Proposition 1.19. $K_*(A_{l,\xi}) = 0$

Proof. By [9, theorem 10.9.6] we have a short exact sequence

$$0 \longrightarrow C_0(]\nu_l, \nu_{l+1}[) \otimes \mathcal{K}(\mathcal{H}) \longrightarrow A_{l,\xi} \xrightarrow{\pi} \mathcal{K}(\mathcal{H}) \longrightarrow 0$$

where $\pi = J_{\xi,\nu_{l+1}}$. As the multiplicity of $J_{\xi,\nu_{l+1}}$ in $\pi_{\xi,\nu_{l+1}}$ is 1 it follows from [8, theorem VI.3.8] that $A_{l,\xi}$ is Morita–equivalent to the algebra A of functions $f \in C_0(]\nu_l, \nu_{l+1}], M_2(\mathbb{C}))$ such that $f(\nu_{l+1}) = \left(\begin{smallmatrix} * & 0 \\ 0 & 0 \end{smallmatrix}\right)$. The six term exact sequence writes

$$0 \longrightarrow K_0(A) \longrightarrow \mathbb{Z} \xrightarrow{\delta} \mathbb{Z} \longrightarrow K_1(A) \longrightarrow 0\,.$$

It suffices to prove that δ is an isomorphism. Let $f \in C_0(]\nu_l, \nu_{l+1}])$ be a positive increasing function such that $f(\nu_{l+1}) = 1$. Then $\delta[1]$ is the class in $K_1\big(C_0(\widetilde{]\nu_l, \nu_{l+1}]})\big)$ of the function $t \mapsto \exp(2i\pi f(t))$. So $\delta[1] = 1$ and δ is an isomorphism. □

We have thus shown that $p_* \in \operatorname{Hom}\big(K_0(C^*(G)), K_0(A)\big)$ is an isomorphism, where $A = C^*_r(G) \oplus (\oplus \mathcal{K}(\mathcal{H}_\pi))$ is the C^*–algebra in the right hand side of the equation (5). All the C^*–algebras in theorem 1.14 are liminar so they satisfy the universal coefficient theorem [19]. In other words we have an exact sequence

$$0 \to \operatorname{Ext}^1_{\mathbb{Z}}\big(K_0(C^*(G)), K_0(A)\big) \longrightarrow KK(C^*(G), A) \longrightarrow \operatorname{Hom}\big(K_0(C^*(G)), K_0(A)\big) \to 0\,.$$

The K–theory groups of the C^*–algebras in 1.14 are free $\mathbb{Z}$–modules. It follows that the Ext group in the previous equation is zero. So the groups $KK(C^*(G), A)$ and $\operatorname{Hom}\big(K_0(C^*(G)), K_0(A)\big)$ are isomorphic. This proves that p induces a KK–equivalence.

2 Range of the Baum–Connes map

Let us first describe explicitly the Baum–Connes isomorphism μ_r. Let us remark that the canonical morphism from $K = \operatorname{Sp}(n) \times \operatorname{Sp}(1)$ to $\operatorname{SO}(\mathfrak{s})$ lifts to a morphism $K \to \operatorname{Spin}(\mathfrak{s})$ because K is simply connected.

To be precise about signs of the indices we now have to choose a decomposition $S = S^+ \oplus S^-$. This one will be fixed along this section. Let V be an euclidian space of even dimension and $(e_1, \ldots, e_{2m})$ an orthonormal basis. The element $\omega = i^m e_1 \cdot \ldots \cdot e_{2m} \in \operatorname{Cliff}_{\mathbb{C}} V$ satisfies $\omega^2 = 1$. So any module E on $\operatorname{Cliff}_{\mathbb{C}} V$ admits a direct sum decomposition $E = E^+ \oplus E^-$

associated to the eigenvalues 1 et -1 of ω. This decomposition only depends on the orientation of the chosen basis. In particular we have a decomposition $S = S^+ \oplus S^-$ and the group $\mathrm{Spin}(2m)$ acts irreducibly on each summand.

Let G be a linear connected semisimple Lie group and let us suppose that G has a compact Cartan subgroup T. Let $K \supset T$ be a maximal compact subgroup in G and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ the Cartan decomposition. Let us choose a positive root system Δ^+ for $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Let $\{\alpha_1, \ldots, \alpha_m\}$ the set of positive noncompact roots. Then we may find [14, Chapter VI, Exercice 5] eigenvectors E_{α_j} and $E_{-\alpha_j}$ so that

$$e_{2j-1} = E_{\alpha_j} + E_{-\alpha_j} \quad \text{and} \quad e_{2j} = i(E_{\alpha_j} - E_{-\alpha_j})$$

are in $\mathfrak{g}$ (and so in $\mathfrak{s}$). Then after normalizing if necessary the family $(e_1, \ldots, e_{2m})$ is an orthonormal basis of $\mathfrak{s}$. So we obtain a decomposition of S that only depends on the set Δ^+. If $\tilde{\Delta}^+$ is another positive root system there exists a unique element w in the Weyl group such that $w\tilde{\Delta}^+ = \Delta^+$. Let $S = \tilde{S}^+ \oplus \tilde{S}^-$ be the decomposition associated to $\tilde{\Delta}^+$. Then

$$\tilde{S}^{\pm} = \begin{cases} S^{\pm} & \text{if } \det w = 1\,, \\ S^{\mp} & \text{if } \det w = -1\,. \end{cases}$$

Choice. Let $G = \mathrm{Sp}(n,1)$. The decomposition $S = S^+ \oplus S^-$ is fixed by the choice of the system Δ^+ defined in the previous section, equation (2).

Let $(\pi, \mathcal{H}_\pi)$ be a unitary irreducible representation of G considered as a $C^*(G)$–module. Let (μ, V) be an irreducible representation of K and $\mathcal{E}_V$ be the completion in the $C^*(G)$–norm of $\Gamma_c(\mathcal{S} \otimes \mathcal{V})$. We have an isomorphism of hermitian spaces

$$\mathcal{E}_V \otimes_\pi \mathcal{H}_\pi \simeq (S \otimes V \otimes \mathcal{H}_\pi)^K \; \Big(= \mathrm{Hom}_K(S \otimes V, \mathcal{H}_\pi) \text{ as } K\text{–module} \Big)\,.$$

Let us recall that $\gamma \colon \mathfrak{s} \to \mathrm{End}\, S$ is the canonical map. The operator $\pi(D_V) = D_V \otimes_\pi 1$ is given by

$$\pi(D_V) = \sum \gamma(x_i) \otimes 1 \otimes \pi(x_i)\,.$$

Notation. The evaluation $m(\pi, \mu)$ of the index $\mathrm{ind}_a\, D_V$ on π is

$$m(\pi, \mu) = \pi_*(\mathrm{ind}_a\, D_V) = \dim \mathrm{Hom}_K(S^+ \otimes V\,,\, \mathcal{H}_\pi) - \dim \mathrm{Hom}_K(S^- \otimes V\,,\, \mathcal{H}_\pi)\,.$$

We will need the following lemma due to Parthasarathy.

Lemma 2.1. *[14, lemme 12.12] Let (μ, V) be an irreducible representation of K and Ω the Casimir operator. Then for any representation $\pi \in \hat{G}$*

$$\pi(D_V)^2 = -\pi(\Omega) + c_\mu\,,$$

with $c_\mu = |\mu + \rho_c|^2 - |\rho|^2$. In particular

$$m_{\pi,\mu} = \pi_*(\mathrm{ind}_a\, D_V) \neq 0 \Longrightarrow \chi_\pi = \chi_{\mu+\rho_c}\,. \tag{6}$$

2.1 Reduced Baum-Connes map

Let us recall that $\hat{M}_{\rm red}$ is the set of irreducible M-representations ξ such that $\pi_{\xi,0}$ is reducible. Let $\mathcal{H}$ be a separable Hilbert space.

Theorem 2.2. *[7] Let $G = \mathrm{Sp}(n,1)$. Then*

$$C_r^*(G) \simeq \left(\underset{\pi\in\mathcal{D}}{\oplus} \mathcal{K}(\mathcal{H}_\pi) \right) \oplus \left(\oplus_{\xi\notin \hat{M}_{red}} A_\xi \right) \oplus \left(\oplus_{\xi\in \hat{M}_{red}} B_\xi \right)$$

where

$$\begin{array}{rcl} A_\xi & = & C_0(i\mathbb{R}^+) \otimes \mathcal{K}(\mathcal{H})\,, \\ B_\xi & = & \left\{ f\in C_0(i\mathbb{R}^+, M_2(\mathbb{C}))\,;\, f(0) = \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right) \right\} \otimes \mathcal{K}(\mathcal{H})\,. \end{array}$$

Lemma 2.3. *[20, Lemma 1.3 and Proposition 1.4]*
1. $K_0(B_\xi) = \mathbb{Z}$ et $K_1(B_\xi) = 0$.
2. $K_0(C_r^*(G))$ is a free $\mathbb{Z}$–module with a set of generators in one to one correspondance with the set $\mathcal{D}\cup\hat{M}_{red}$.

Let $\xi\in\hat{M}_{\rm red}$ and $\pi^\pm$ be the two subrepresentations of $\pi_{\xi,0}$. The maps $\pi^\pm \colon B_\xi \to \mathcal{K}(\mathcal{H}_{\pi^\pm})$ induce isomorphisms in K–theory. We will write $[\pi^\pm]$ for the preimage under this isomorphism of the generator 1 of $K_0(\mathcal{K}(\mathcal{H}_{\pi^\pm})) = \mathbb{Z}$. Similarly for a given discrete series $\pi\in\mathcal{D}$ we write $[\pi]$ the preimage under π_* in $K_0(C_r^*(G))$ of the generator 1 of $K_0(\mathcal{K}(\mathcal{H}_\pi))$.

In view of equation (6) we will need to label the irreducible admissible representations with a given integral infinitesimal character. We use in this paper the labelling defined by Baldoni Silva and Kraljević in [2]. Let χ be a regular integral infinitesimal character. For any $j = 0,\dots,n$ let C_j be the positive root system in Δ defined by the closed Weyl chamber

$$\tilde{C}_j = \left\{ \sum a_i\epsilon_i\,;\, a_1 \geq \cdots \geq a_j \geq a_{n+1} \geq a_{j+1} \cdots \geq a_n \geq 0 \right\}\,.$$

For example Δ^+ corresponds to the chamber C_n. For any $j = 0,\dots,n$ there exists a unique $\lambda_j \in \tilde{C}_j$ such that $\chi_{\lambda_j} = \chi$. The discrete series with infinitesimal character χ are those whose Harish–Chandra parameter are the λ_j's. We denote by $\pi_j(\chi)$ (or π_{λ_j}) the discrete series with Harish-Chandra parameter λ_j.

Up to equivalence the principal series with infinitesimal character χ have a Langlands parameter $\gamma = \Lambda_{\xi,\nu} = \sum a_i e_i$ satisfying $\chi = \chi_\gamma$ and

$$\mathrm{Re}\,(a_1+a_2) > 0 \quad \text{or} \quad \left(\mathrm{Re}\,(a_1+a_2) = 0 \text{ and } \mathrm{Im}\,(a_1+a_2) \geq 0\right). \tag{7}$$

For any (i,j) such hat $0 \leq i < j \leq n+1$ let P_{ij} be the system of positive roots in Φ corresponding to the Weyl chamber

$$\begin{array}{rl} \tilde{P}_{i,j} = \{ \sum a_i e_i\,;\, a_3 \geq \cdots & \geq a_{i+2} \geq a_1 \geq a_{i+3} \geq \cdots \\ & \geq a_{j+1} \geq a_2 \geq a_{j+2} \geq \cdots \geq a_{n+1} \geq 0 \}\,, \end{array}$$

and for any (i,j) such that $0 \leq i \leq n$, $n+1 \leq j \leq 2n-i$ let P_{ij} be the system corresponding to the chamber

$$\begin{array}{rl} \tilde{P}_{i,j} = \{ \sum a_i e_i\,;\, a_3 \geq \cdots & \geq a_{i+2} \geq a_1 \geq a_{i+3} \geq \cdots \\ & \geq a_{2n+2-j} \geq -a_2 \geq a_{2n+3-j} \geq \cdots \geq a_{n+1} \geq 0 \}\,. \end{array}$$

The positive root systems P_{ij} are chosen so that if γ satisfies the equation (7) and $\chi_\gamma = \chi$ there exists a unique couple (i,j) such that $\gamma \in \tilde{P}_{ij}$. Let us denote by $\pi_{ij}(\chi)$ (or $\pi(\gamma)$) the (unique) principal series with infinitesimal character χ and Langlands parameter $\gamma \in \tilde{P}_{ij}$. Let $J_{ij}(\chi)$ be the Langlands quotient of $\pi_{ij}(\chi)$.

Thanks to the Langlands classification theorem the admissible irreducible representations of $\mathrm{Sp}(n,1)$ with infinitesimal character χ are the Langlands quotients $J_{ij}(\chi)$ and the discrete series $\pi_j(\chi)$.

Let χ be a singular integral infinitesimal character and $\gamma = \sum_{j=1}^{n+1} a_j e_j$ such that $\chi_\gamma = \chi$. Then $a_j \in \mathbb{Z}$ and

$$a_1 > a_2\,, \quad a_1 \geq -a_2\,, \quad a_3 > \cdots > a_{n+1} > 0\,. \tag{8}$$

This implies that γ is contained in at most two walls.

Lemma 2.4. *If γ is contained in two walls then $\pi(\gamma)$ is irreducible.*

In fact if γ is contained in two walls there are no $\gamma' \neq \gamma$ such that $\chi_\gamma = \chi_{\gamma'}$ and satisfying equation (8).

Let us suppose that γ is contained in exactly one wall. Let $\lambda \in \mathfrak{t}^*_{\mathbb{C}}$ such that $\chi = \chi_\lambda$. There exists a unique $i \in \{0,\ldots,n-1\}$ such that λ is contained in the wall between $\tilde{C}_{i-1}$ and $\tilde{C}_i$. Let ρ^j be the half sum of positive roots of the system C_j $(j = 0,\ldots,n)$. Then $\lambda + \rho^{i-1}$ and $\lambda + \rho^i$ are regular with respect to C_{i-1} and C_i respectively. Let $\Psi_\lambda^{\lambda'}$ be the Zuckerman translation functor. Here we use the notations of Knapp's book [14, Chapter X, section 9]. To such a λ we associate two limits of discrete series determined by the choice between C_{i-1} and C_i. To be more precise the limits of discrete series π_λ^- et π_λ^+ are defined by

$$\begin{aligned}\pi_\lambda^+ &= \pi^+(\chi) = \Psi_\lambda^{\lambda+\rho^i}(\pi_{\lambda+\rho^i})\\ \pi_\lambda^- &= \pi^-(\chi) = \Psi_\lambda^{\lambda+\rho^{i-1}}(\pi_{\lambda+\rho^{i-1}})\,.\end{aligned}$$

Any limit of discrete series is obtained by this way. Let $\mu \in \hat{K}$ and ρ_c the half sum of compact positive roots in Δ^+. Let $\chi = \chi_{\mu+\rho_c}$. If χ is regular let $\pi_i(\chi)$ $(i = 0,\ldots,n)$ be the discrete series with infinitesimal character χ. If χ is singular then there exists two limits of discrete series with infinitesimal character χ denoted $\pi^+(\chi)$ et $\pi^-(\chi)$ in accordance with the choice made just before.

Theorem 2.5. *Let $\mu \in \hat{K}$ and $\chi = \chi_{\mu+\rho_c}$. If χ is regular there exists a unique $i \in \{0,\ldots,n\}$ such that $\mu + \rho_c \in \tilde{C}_i$. Then we have in $K_0(C_r^*(G))$,*

$$Ind_a\, D_\mu = (-1)^{n-i}[\pi_i(\chi)]\,.$$

If χ is singular, there exists a unique i such that $\mu + \rho_c \in \tilde{C}_{i-1} \cap \tilde{C}_i$. Then in $K_0(C_r^*(G))$

$$Ind_a\, D_\mu = (-1)^{n-i}[\pi^+(\chi)]\,.$$

The end of this subsection is devoted to the proof of this theorem.

Thanks to Parthasarathy's lemma (Lemma 2.1) the irreducible tempered representations π of G for which $\pi(D_\mu)$ has a non–zero kernel satisfy $\chi_\pi = \chi$. If χ is regular it suffices to compute D_μ on the discrete series $\pi_j(\chi)$. After Atiyah–Schmid's theorem [1] there exists a unique discrete series π such that $m(\pi,\mu) \neq 0$ and this representation is the discrete series with Harish–Chandra parameter $\mu + \rho_c$. Moreover if $\pi = \pi_{\mu+\rho_c} = \pi_i(\chi)$ we have $m(\pi,\mu) =$

$\pi_* \operatorname{Ind}_a D_\mu = 1$ whenever the chosen positive root system defining the decomposition of S is such that $\mu + \rho_c$ is dominant. The conclusion in the regular case follows.

Let us assume now that χ is singular. Applying again Parthasarathy's lemma we see that it suffices to compute the index on the unique component B_ξ of $C_r^*(G)$ such that $\pi_{\xi,0}$ is the direct sum of the limits of discrete series $\pi^+(\chi)$ and $\pi^-(\chi)$. By proposition 2.8 (this proposition is a general fact of representation theory of semi–simple groups - see below) we have to compute the character of a limit of discrete series. Let us recall that if λ is a linear form on $\mathfrak{t}$ such that $\lambda \in \tilde{C}_{j-1} \cap \tilde{C}_j$, the corresponding limit of discrete series $\pi^-(\lambda)$ (resp. $\pi^+(\lambda)$) is constructed by means of the Zuckermann translation functor by chosing the positive root system $\Delta(\mathfrak{t}_\mathbb{C}, \mathfrak{g}_\mathbb{C})$ corresponding to the Weyl chamber $\tilde{C}_{j-1}$ (resp. $\tilde{C}_j$). In other words

$$\pi^+(\lambda) = \Psi_\lambda^{\lambda'}(\pi_{\lambda'}) ,$$

with λ' integral regular with respect to C_{j-1} (resp. C_j) and $\pi_{\lambda'}$ is the discrete series with Harish-Chandra parameter λ' or equivalently the unique discrete series π such that $\pi_*(\mathrm{ind_a} D^+_{\mu'})$ $\neq 0$ and $\mu' = \lambda' - \rho_c$. We know by [14, Theorem 12.7] that the character of a discrete series π' with parameter λ' satisfies $\Delta_T \theta_{\pi'}|_{T'} = \sum_{w \in W_K} \det w \, e^{w\lambda'}$. Now using [14, Proposition 10.44] (effect of the Zuckermann functor on characters) we obtain $\Delta_T \theta_\pi|_{T'} = \sum_{w \in W_K} \det w \, e^{w\lambda}$. As we have already said these formulas depend on the choice of a positive root system such that λ' is dominant. Thus the proposition 2.8 gives the desired result.

2.2 Full Baum-Connes map

Let $\mu \in \hat{K}$ and $\chi = \chi_{\mu+\rho_c}$. If χ is regular there exists a unique $i(\chi) \in \{0, \ldots, n\}$ such that $\mu + \rho_c \in C_i$. There also exists a unique $\gamma \in P_{0,1}$ such that $\chi_\gamma = \chi$. If χ is singular there exists a unique $i(\chi) \in \{0, \ldots, n\}$ such that $\mu + \rho_c \in C_{i-1} \cap C_i$. There also exists a unique $\gamma \in P_{0,i-1} \cap P_{0,i}$ such that $\chi_\gamma = \chi$. Let $\alpha_i = e_i - e_{i+1}$ $(i = 1, \ldots, n)$ and $\alpha_{n+1} = 2e_{n+1}$ be the simple roots of Φ^+ and let δ be the half sum of positive roots in Φ^+. We define $p(\chi) \in \{0, \ldots, n+1\}$ to be the smallest integer p such that

$$\langle \gamma - \delta \, , \, \alpha \rangle = 0 \quad \forall \alpha = \alpha_{p+1}, \ldots, \alpha_{n+1} \, .$$

The following lemma is a consequence of the theorems [2, Theorem 3.6,4.2] and [3, Theorem 7.1].

Lemma 2.6. *If χ is regular the « isolated » series with infinitesimal character χ are the Langlands quotients $J_{j,j+1}(\chi)$ with $p(\chi) \leq j \leq n-2$.*

If χ is singular there exists at most one isolated series with infinitesimal character χ. Moreover there exists an isolated series $J(\chi)$ with infinitesimal character χ if and only if $i(\chi) \geq p(\chi)$.

Let π be an « isolated » series. We denote by $[\pi]$ the generator of $K_0(C^*(G))$ given by Theorem 1.14.

Theorem 2.7. *Let $\mu \in \hat{K}$ such that $\chi = \chi_{\mu+\rho_c}$ Let $i = i(\chi)$.*

If χ is regular we have in $K_0(C^*(G))$,

$$\operatorname{Ind}_a(D^+_\mu) = (-1)^{n-i} \left([\pi_i(\chi)] + \sum_{j=p(\chi)}^{\min(i,n-2)} [J_j(\chi)] \right) .$$

If χ is singular

$$\mathrm{Ind}_a(D_\mu^+) = \begin{cases} (-1)^{n-i}\left([\pi^+(\chi)] + [J(\chi)]\right) & \textit{if } i(\chi) \geq p(\chi) \\ (-1)^{n-i}[\pi^+(\chi)] & \textit{otherwise.} \end{cases}$$

The end of this subsection is devoted to the proof of this theorem. For $\mu \in \hat{K}$ and π an « isolated » series we want to compute

$$m(\pi,\mu) = \dim \mathrm{Hom}_K(V_\mu \otimes S^+, H_\pi^{(K)}) - \dim \mathrm{Hom}_K(V_\mu \otimes S^-, H_\pi^{(K)}) .$$

This will determine an equality

$$\mathrm{Ind}_a D_\mu^+ = \sum_\pi m(\pi,\mu)[\pi] .$$

We first recall some basic facts about characters of admissible representations of semisimple groups. Let G be a connected semisimple linear Lie group and K a maximal compact subgroup. Let us suppose that K has a Cartan subgroup T that is a Cartan subgroup in G. Let Φ be the root system of $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. We take a positive root system Φ^+ and we denote by Φ_c^+ (resp.Φ_n^+) the set of positive compact roots (resp. nonompact roots). For any unitary representation (η, V) of K we denote by $\mathrm{ch}(\eta)$ its character. Let us suppose that π is an admissible representation of G and let us write

$$\pi|_K = \sum_{\eta \in \hat{K}} n_\eta V_\eta .$$

By [14, Lemma 12.8] the series $\sum n_\eta \mathrm{ch}(\eta)$ converges to a distribution $\mathrm{ch}(\pi)$ of K .

Now using the fact that the representations S^+ and S^- are selfadjoint when q is even and dual to each other if q is odd we find that if we write

$$\mathrm{ch}(\pi)(\mathrm{ch}(S^+) - \mathrm{ch}(S^-)) = \sum a(\pi,\eta)\mathrm{ch}(\eta)$$

then

$$m(\pi,\eta) = (-1)^q a(\pi,\eta) \text{ with } q = \tfrac{1}{2}\dim(G/K)\,(= 2n \text{ for } G = \mathrm{Sp}(n,1)) .$$

Let us recall that θ_π is the function of G' that defines the character of π. We then have $\mathrm{ch}(\pi) = \theta_\pi|_{G' \cap K}$. Let $T' = G' \cap T$. We have

$$(\mathrm{ch}(S^+) - \mathrm{ch}(S^-))|_{T'} = \prod_{\beta \in \Phi_n^+} (e^{\beta/2} - e^{-\beta/2}) ,$$

and thanks to the Weyl formula

$$\Delta_T^K \mathrm{ch}\mu = \sum_{w \in W_K} \det w \, e^{w(\mu+\rho_c)}$$

where $\Delta_T^K = \prod_{\beta \in \Phi_c^+}(e^{\beta/2} - e^{-\beta/2})$. Let $\Delta_T = \prod_{\beta \in \Phi^+}(e^{\beta/2} - e^{-\beta/2})$. We have

$$\begin{aligned} \Delta_T \theta_\pi|_{T'} &= (\mathrm{ch}(S^+) - \mathrm{ch}(S^-))\Delta_T^K \theta_\pi|_{T'} \\ &= \textstyle\sum_\mu a(\pi,\mu)\Delta_T^K \mathrm{ch}(\mu) \\ &= \textstyle\sum_{\mu,w} \det w \, a(\pi,\mu) e^{w(\mu+\rho_c)} \end{aligned}$$

In particular

Proposition 2.8. *[1] To compute $m(\pi,\mu)$ it suffices to know the coefficient of $e^{\mu+\rho_c}$ in $\Delta_T\theta_\pi|_{T'}$.*

We now come back to the group $G = \mathrm{Sp}(n,1)$. The next proposition is an immediate corollary of Theorem 1.7.

Proposition 2.9. *If $\pi = \pi_{\xi,\nu}$ is a principal series then $\theta_\pi|_{T'} = 0$.*

We are now ready to study the case where π is an « isolated » series.

Lemma 2.10. *[14, Lemma 12.9] Let ρ_n^i the half sum of positive noncompact roots of the system of positive roots C_i defined before. For any $0 \leq i \leq n$ we have*

$$S = \bigoplus_{i=0}^{n} V_{\rho_n^i} \quad ; \quad S^+ = \bigoplus_{i\ even} V_{\rho_n^{n-i}} \quad ; \quad S^- = \bigoplus_{i\ odd} V_{\rho_n^{n-i}} \quad .$$

We deduce immediately the following proposition.

Proposition 2.11. *If $\pi = 1_G$ is the trivial representation then*

$$m(\pi,\mu) = \begin{cases} 1 & \text{if } \mu = \rho_n^{n-i} \text{ and } i \text{ is even,} \\ -1 & \text{if } \mu = \rho_n^{n-i} \text{ and } i \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\chi = \chi_\pi$ be the infinitesimal character of π. It is integral. Baldoni–Silva and Kraljević [2] give an expression of the characters of the principal series in function of the irreducible character of G. More precisely they give an explicit formula

$$\theta_{\pi_{i,j}(\chi)} = \sum_{\pi \in \{J_{k,l}(\chi)\} \cup \{\pi_k(\chi)\}} n_{\pi_{i,j}(\chi),\pi}\theta_\pi \,. \tag{9}$$

where the sum runs over all admissible irreducible representations with infinitesimal character χ. We want to deduce a formula of the form

$$\theta_\pi = \sum_{\pi_{\xi,\nu},\, \chi_{\pi_{\xi,\nu}}=\chi} n'_{\pi,\pi_{\xi,\nu}}\theta_{\pi_{\xi,\nu}} + \sum_{\pi' \in \mathcal{D}\cup\mathcal{L},\, \chi_{\pi'}=\chi} n'_{\pi,\pi'}\theta_{\pi'} \,. \tag{10}$$

Once proved it would then suffice to apply Proposition 2.9 and the corresponding result for the discrete series or the limits of discrete series (Theorem 2.5) to obtain $m(\pi,\mu)$.

This combinatorial problem can be rephrased in these terms. We define an oriented graph Γ_χ^n whose vertices are the irreducible representations with infinitesimal character χ. We put an egde from J to π' if J is the Langlands quotient of a principal series say π_J and $\pi' \neq J$ is a subquotient of π_J. The number of such edges is $n_{\pi_J,\pi'}$. Let $\pi' = \pi_i(\chi)$ be the discrete series corresponding to μ (by Theorem 2.5) if χ is regular and $\pi' = \pi^+(\chi)$ if χ is singular. The number $n'_{J,\pi'}$ is the difference between the number of paths in Γ_χ^n from J to π' with even length and the number of such paths with odd length.

Let us first assume that χ is regular. The following lemma is an immediate corollary of [2, Theorem 3.6].

Lemma 2.12. *Let j such that $\pi = J_{j,j+1}(\chi)$. Let $\pi' = \pi_i(\chi)$.*
1. If $j > i$ then there is no path from π to π' in Γ^n_χ.
2. If $j \leq i$, any path from π to π' is contained in a subgraph of Γ^n_χ isomorphic to Γ^{n-j}_0 (associated to the group $\mathrm{Sp}(n-j,1)$ and the trivial infinitesimal character). Moreover in this subgraph π is the point corresponding to the trivial representation in Γ^{n-j}_0 and π' is the point corresponding to the discrete series $\pi_{i-j}(0)$ of $\mathrm{Sp}(n-j,1)$.

The regular case in Theorem 2.7 then follows from Proposition 2.11 and Theorem 2.5.

It remains to treat the case where χ is singular. Using [2, Theorem 4.2] it is easy to show the following lemma.

Lemma 2.13. *There is only one path in the graph Γ^n_χ from J to $\pi' = \pi^+(\chi)$ and its length is $2n-2i+2$ and so is even.*

The singular case in Theorem 2.7 then follows from Theorem 2.5.

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