Hypoellipticity and cohomological induction

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Abstract

Let $Y = G/L$ be a flag manifold for a reductive G and K a maximal compact subgroup of G. We define here an equivariant differential operator on $G/L \cap K$ playing the role of the Dolbeault Laplacian for the complex manifold G/L, using a distrbution transverse to the fibers of $G/L \cap K \to G/L$ and satisfying the Hörmander condition. We prove here that this operator is (surprisingly but hopefully) not maximal hypoelliptic.

Introduction

There are two challenging problems in representation theory of Lie groups. The first one is to classify unitary representations for large classes of Lie groups. Connected nilpotent Lie groups form such a class, and Kirillov established, for any connected nilpotent Lie group, a bijective correspondance between the set of coadjoint orbits and the set of (equivalence classes of) unitary irreducible representations of the group. This approach lead to the second problem : to realize unitary representations geometrically. These two problems are still open for reductive groups, but the technique of coadjoint orbits is a constant source of inspiration. For reductive groups there are three kind of orbits : the hyperbolic orbits, the elliptic orbits, and the nilpotent orbits. The hyperbolic orbits lead to the theory of parabolic induction and Knapp-Stein intertwining operators. This is appropriate to construct unitary representations that are weakly contained in the regular representation. The elliptic orbits are related with the theory of cohomological induction and the geometry of flag manifolds. The study of nilpotent orbits lead to the theory of unipotent representations. We are concerned here with the geometry of flag manifolds and we use the theory of coadjoint orbits for nilpotent Lie groups to handle regularity problems of differential operators on flag manifolds.

Let G be a reductive Lie group and Y be a flag manifold for G. The G-space Y is a complex manifold with an equivariant complex structure, and is a homogeneous space of the form G/L, where the Lie subgroup L is reductive but don't need to be compact. A representation χ of L is chosen, and the usual Dolbeault complex is twisted by χ. The smooth cohomology $H^*(\overline{\partial}_\chi)$ of this complex is proved by H. W. Wong [Won95] to be a Fréchet representation of G whose underlying Harish-Chandra module is isomorphic to the cohomologically induced representation $R(\chi)$. The proof of H. W. Wong uses the double fibration $G/L \leftarrow G/L \cap K \to G/K$, where the group K is a maximal compact subgroup of G. One conjecture that if χ is a unipotent unitary representation of L, whatever it means, then the representation $H^*(\overline{\partial}_\chi)$ is unitarizable. However, as a Fréchet space it can not carry a unitary structure. In the best case, when L is compact, one choose a G-invariant hermitian metric on Y and then consider two objects : the Hilbert space of square integrable sections of the twisted Dolbeault complex, and the Dolbeault laplacian $\overline{\Box}_\chi = \overline{\partial}_\chi \overline{\partial}_\chi^* + \overline{\partial}_\chi^* \overline{\partial}_\chi$. This differential operator is elliptic and is a selfadjoint operator on the Hilbert space. Its L^2-kernel is then proved to be a unitary representation that infinitesimally isomorphic to the Fréchet representation. Such representations are sums of discete series [AS77],[CM82]. In the general case, necessary to find other representations, the flag manifold does not carry any G-invariant hermitian metric. A positive metric is then defined in [RSW83] to define the Hilbert space, and I proved in full generality [Pru06] that this Hilbert space is a continuous G-module. The proof again uses the double fibration considered by Wong. To replace the G-invariant selfadjoint operator, an invariant non-positive form on Y is defined [RSW83],[BKZ92]. It is used to define the adjoint $\overline{\partial}_{\chi,\mathrm{inv}}^*$ and the harmonic space $\ker \overline{\partial}_\chi \cap \ker \overline{\partial}_\chi^*$. The point is that the invariant operator $\overline{\partial}_\chi \overline{\partial}_{\chi,\mathrm{inv}}^* + \overline{\partial}_{\chi,\mathrm{inv}}^* \overline{\partial}_\chi$ does not satisfy any regularity condition such as ellipticity and can not be used.

We propose here a new invariant operator, defined via the fibration $\pi_L \colon G/L \cap K \to G/L$ and study its regularity properties as an operator on $G/L \cap K$. We first define a distribution transverse to the fibers that satisfies the Hörmander's condition. It is used to pullback the Dolbeault operator also denoted by $\overline{\partial}$. The manifold $G/L \cap K$ has a G-invariant positive metric defined by the Killing form, and we can use it to define the formal adjoint $\overline{\partial}^*$ of the pullback of the Dolbeault operator. We then define $\overline{\Box} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$. We first show that on sections constant along the fibers, this operator equals (up to an operator of lower order) the Hörmander Laplacian which is known to be maximal hypoelliptic. We next show that on the whole space of sections over $G/L \cap K$ the operator $\overline{\Box}$ is not maximal hypoelliptic. To prove this we provide the tangent space of $G/L \cap K$ with a nilpotent algebra structure, canonically associated to the fibration π_L, and find a non trivial irreducible representation σ of the associated connected nilpotent Lie group such that $\sigma(\overline{\Box})$ is not injective on the space of smooth vectors of σ. Actually this turns out to be the case for many representations.

The representation χ would have been of interest for the (more delicate) questions of positivity for instance but does not come into questions of regularity ; we then use the usual Dolbeault complex.

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1 The Dolbeault Laplacian

1.1 Definition

Let $Y = G/L$ be a flag manifold for a reductive Lie group G. This means that Y is an open orbit in a flag manifold $G^{\mathbb{C}}/Q$, where $G^{\mathbb{C}}$ is the complexified Lie group of G

and Q is a parabolic subgroup of $G^{\mathbb{C}}$; we also require that Y admits a G-invariant measure. We note $\mathfrak{g}_0$ the Lie algebra of G, and $\mathfrak{g}$ its complexification and use the same convention with other real and complex Lie algebras and spaces. Then Y has an equivariant complex structure. Choices of a maximal compact subgroup K of G and of a base point $y_0 \in Y$ can be made such that the reductive group $L = \mathrm{Stab}_G(y_0)$ is the centralizer of a compact torus with Lie algebra $\mathfrak{t}_0' \in \mathfrak{g}_0$, $L^{\mathbb{C}}$ is the Levi part of Q and $K/L \cap K$ is a maximal compact complex submanifold of Y. The parabolic algebra $\mathfrak{q}$ has a decompostion $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, and $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{u} \oplus \overline{\mathfrak{u}}$ with $X \mapsto \overline{X}$ is the conjugaison associated to the real form G of $G^{\mathbb{C}}$. The space $\mathfrak{u}$ is L-isomorphic to the antiholomorphic tangent space $T_e^{0,1}G/L$. Note that the connected reductive subgroup L need not to be compact, so that Y does not have a G-invariant Riemannian metric in general.

The manifold Y has a G-invariant complex structure : this means that the De Rham operator d writes $d = \partial + \overline{\partial}$, where $\partial\colon \wedge^{p,q}TY_{\mathbb{C}} \to \wedge^{p+1,q}TY_{\mathbb{C}}$ and $\overline{\partial}\colon \wedge^{p,q}TY_{\mathbb{C}} \to \wedge^{p,q+1}TY_{\mathbb{C}}$ are G-equivariant operators. The restriction to $\wedge^{0,*}TY = \wedge^* \mathfrak{u}$ of the operator $\overline{\partial}$ is called the Dolbeault operator . The manifold $Z = G/L \cap K$ fibers over Y and the group G acts on it properly. It then admits a G-invariant Riemannian metric. We define the horizontal space at a point $z \in Z$ to be the orthocomplement E_z of the space F_z tangent at z to the fiber trough z.

We then have a connexion E on the fibration π_L which enables to pullback the Dolbeault operator.

Proposition 1. *Let Y be a complex manifold with G-invariant complex structure and $\pi\colon Z \to Y$ an equivariant fibration, with fiber F. We suppose that the exact sequence*

$$TF \to TZ \to \pi^* TY$$

has an equivariant splitting. Let $p_*^{0,1}$ be the transposed map of this splitting followed by the projection p_* to the (pullback of the) antiholomorphic tangent space $\pi^* T^{0,1}Y$. Then there exists a unique operator $\overline{\partial}'$ on Z satisfying the following conditions.

$$\overline{\partial}'(\pi^*\omega) = \pi^*(\overline{\partial}\omega) \tag{1}$$

$$\overline{\partial}'(f\pi^*\omega) = p_*^{0,1} df \wedge (\pi^*\omega) + f\pi^*(\overline{\partial}\omega) \tag{2}$$

The operator $\overline{\partial}'$ will be denoted $\overline{\partial}$ when no confusion arises.

Proof. We have to check that, for any $f \in C^\infty(Z)$, any $g \in C^i nfty(Y)$ non zero, and any $\omega \in \Gamma(Y, \wedge^* T^{0,1}Y)$, we have $\overline{\partial}'(f\pi^* g \pi^*(g^{-1}\omega)) = \overline{\partial}'(f\pi^*\omega)$. Now,

$$\begin{aligned}
\overline{\partial}'(f\pi^* g\pi^*(g^{-1}\omega)) &= p_*0,1d(f\pi^* g) \wedge \pi^*(g^{-1}\omega) + f\pi^* g\pi^*(\overline{\partial}(g^{-1}\omega)) \\
&= p_*0,1(df)\pi^* g\pi^*(g^{-1}) \wedge \pi^*(\omega) + fp_*0,1d(\pi^* g)\pi^*(g^{-1}) \wedge \pi^*(\omega) \\
&\quad + f\pi^* g\pi^* \overline{\partial}(g^{-1})\pi^*\omega + f\pi^* g\pi^*(g^{-1})\pi^*(\overline{\partial}\omega) \\
&= p_*0,1df \wedge (\pi^*\omega) + f\pi^*(\overline{\partial}\omega) + f\pi^*(g^{-1}\overline{\partial} g + g\overline{\partial}(g^{-1})) \wedge \omega \\
&= \overline{\partial}'(f\pi^*\omega)\,.
\end{aligned}$$

□

The action of G on $G/L \cap K$ is proper. In particular, this G-space admits a G-invariant Riemannian metric. As usual the choice of such a metric enables to define a bilinear pairing $(\, , \,)$ between the space of forms with compact supports and the space of forms. The $*$-operator is then given by $(\alpha, \beta) d\mathrm{vol} = \alpha \wedge (*\beta)$. We then define the adjoint of the pullbacked Dolbeault operator (on homogeneous forms) by

$$\overline{\partial}^* \omega = (-1)^{|\omega|} (*\overline{\partial}*) \omega \,.$$

It remains to define the Dolbeault Laplacian by

$$\overline{\Box} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial} \,.$$

This operator is G-equivariant by construction. The question is : can we build an algebra of pseudodifferential operators on which the Dolbeault laplacian admits a parametrix ? The result we prove here gives a negative answer to that question.

1.2 Structure of the transerve subbundle

The Dolbeault Laplacian is clearly not elliptic. To study more involved regularity properties of this operator, we will need detailed information on the bundle E. We now investigate the structure of this bundle E.

Definition 2. *A subbundle E of the tangent space TZ is a 2-step* bracket generating *subbundle if for any point $p \in Z$, the space $[X, Y](p) \mod E_p$, with X and Y running over sections of E,is the whole space T_pZ/E_p. In particular the bundle homomorphism*

$$[\, , \,]_0 \colon \bigwedge{}^2 E \longrightarrow TZ/E \,, \tag{3}$$

induced by the barcket $[\, , \,]$ of vectors fields, is onto. We say that E satisfies the Hörmander condition at order 2*.*

Lemma 3. *The subbundle E of the tangent bundle satisfies the Hörmander condition at order* 2*.*

Proof. Without lost of generality we may assume that $\mathfrak{g}$ is simple. It is enough to prove that $\mathfrak{s} = \mathfrak{u} \oplus \overline{\mathfrak{u}} + [\mathfrak{u}, \overline{\mathfrak{u}}]$ is a non zero ideal of $\mathfrak{g}$. As $\mathfrak{q}$ is a parabolic subalgebra, $\mathfrak{u}$ and $\overline{\mathfrak{u}}$ are sums of root spaces. Moreover $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{u} \oplus \overline{\mathfrak{u}}$, so we get

$$[\mathfrak{u}, \overline{\mathfrak{u}}] = ([\mathfrak{u}, \overline{\mathfrak{u}}] \cap \mathfrak{l}) \oplus ([\mathfrak{u}, \overline{\mathfrak{u}}] \cap \mathfrak{u}) \oplus ([\mathfrak{u}, \overline{\mathfrak{u}}] \cap \overline{\mathfrak{u}}) \,.$$

Using this one checks that $[\mathfrak{u} \oplus \overline{\mathfrak{u}}, [\mathfrak{u}, \overline{\mathfrak{u}}]] \subset \mathfrak{s}$. Let $X, X' \in [\mathfrak{u}, \overline{\mathfrak{u}}]$. We write $X = X_l + X_u + X_{\overline{u}}$ thanks to the preceding equation, and $X' = [X'_u, X'_{\overline{u}}]$. This gives : $[X, X'] = [[X_l, X'_u], X'_{\overline{u}}] + [X'_u, [X_l, X'_{\overline{u}}]] + X''$ with $X'' \in \mathfrak{s}$. So $[X, X'] \in \mathfrak{s}$ and $\mathfrak{s}$ is a subalgebra of $\mathfrak{g}$. It is also clearly stable by $\mathfrak{l}$. ☐

We now state a more precise result when G is the group $U(p, q)$ and $L = U(p_1) \times U(p_2, q)$ with $p_1 + p_2 = p$.

Lemma 4. *There exists a sequence $\Gamma = (\gamma_1, \ldots, \gamma_r)$ of roots in $\Delta(\mathfrak{l} \cap \mathfrak{p})$, such that, for any $\alpha \in \Delta(\mathfrak{u})$ there exist at most one $1 \leq i \leq r$ and $\beta \in \Delta(\mathfrak{u})$ such that $\alpha + \gamma_i = \beta$.*

Proof. The roots in Δ are $e_i - e_j$ and

$$\Delta(\mathfrak{l}\cap\mathfrak{p}) = \{e_i - e_j\,;p_1 < i \le p < j \le p+q\}$$
$$\Delta(\mathfrak{u}) = \{e_i - e_j\,;1 \le i \le p_1 < j \le p+q\}\,.$$

Set $r = min\{p_1, q\}$ (the real rank of the noncompact semi simple part of $\mathfrak{l}$) and let $\Gamma = (\gamma_i)$ be any set of strongly orthogonal roots. For exemple, one may take $\gamma_i = e_{p_1+i} - e_{p+q-i}$. The result follows easily. In fact, if $\alpha = e_i - e_j \in \Delta(\mathfrak{u})$ then the only $\beta = e_k - e_l$ that may work are those with $k = i$ or $l = j$, and only one of them can lies in Γ. $\square$

1.3 Statement of the main result

Let us precise now the regularity property of differential operators we want to investigate. Let $X_1, \ldots, X_k$ be vector fields on a neighborhood V of a point $x_0 \in \mathbb{R}^n$, and let E_{x_0} be the subspace of $\mathbb{R}^n$ generated by the vectors $X_i(x_0)$. We also assume that vectors $[X_i, X_j](x_0) \mod (E_{x_0})$ generate the vector space $\mathbb{R}^n/E_{x_0}$. The space of operators of order less than m is the space of operator P that can be written in the form

$$P = \sum_{|\alpha|\le m} a_\alpha(x) X^\alpha\,, \quad X^\alpha = X_1^{\alpha_1}\cdots X_k^{\alpha_k}, \tag{4}$$

where the coefficient $a_\alpha(x)$ are smooth functions of the variable x on V.

Definition 5. *[HN85] A differential operator P of order m is maximal hypoelliptic at x_0, if there exists a neighborhood V of x_0 and a constant C so that for all $u \in C_c^\infty(V)$,*

$$\sum_{|\alpha|\le m} \|X^\alpha u\|_{L^2} \le C\big(\|u\|_{L^2} + \|Pu\|_{L^2}\big)\,.$$

Maximal hypoellipticity of an operator P implies that P is hypoelliptic, i.e.

$$Pu \text{ smooth} \quad \Rightarrow \quad u \text{ smooth}\,.$$

The principal E-symbol is by definition $p = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$. We will use the sign "$\simeq$" to say that two operators have the same principal E-symbol. The following result is well known (see [HN85]).

Proposition 6. *The Hörmander Laplacian $\sum_i X_i^2$ is maximal hypoelliptic.*

We choose here the metric given by the Killing form B. More precisely, the metric is defined at the origin by

$$\langle X, Y\rangle = -B(X, \theta(\overline{Y}))\,.$$

This form is definite positive on $\mathfrak{g}$ and is $\mathfrak{k}$-invariant. The tangent spaces $T_eY \simeq \mathfrak{u}\oplus\overline{\mathfrak{u}}$ and $T_eZ \simeq \mathfrak{u}\oplus\overline{\mathfrak{u}}\oplus(\mathfrak{l}\cap\mathfrak{p})$ are provided with this hermitian metric.

Theorem 7. *Let $G = \mathrm{SU}(p,q)$ and $L = \mathrm{S}(\mathrm{U}(p_1)\times\mathrm{U}(p_2,q))$, with $p_1 + p_2 = p$. The Laplacian $\overline{\square}$ is not maximal hypoelliptic at the origin $eL\cap K$.*

One may conjecture this this result is true for any semisimple Lie group and flag manifold. The exposition of the proof is intended to make clear that only the lemma 4 as to be generalized. So let G be a semisimple Lie group with a compact Cartan subaglra, and G/L be a flag manifold for G. This assumption on the Cartan subalgebra makes less technical the computation of the principal E-symbol , but we should proceed without it.

The next section is devoted to the proof of the theorem 7. To prepare the proof we compute here the local expression of the principal E-symbol of this operator. The Cartan subalgebra being compact, we may suppose that $\mathfrak{h}_0 \supset \mathfrak{t}_0$, so that

$$\mathfrak{t}'_0 \subset \mathfrak{t}_0 \subset \mathfrak{l}_0 \cap \mathfrak{k}_0 \subset \mathfrak{l}_0 \subset \mathfrak{g}_0 \,.$$

Let Δ be the root system of the pair $(\mathfrak{g}, \mathfrak{t})$ All roots of the root system $\Delta(\mathfrak{g}, \mathfrak{t})$ being compact or non compact, it makes sens to define $\Delta(\mathfrak{u} \cap \mathfrak{k})$ and $\Delta(\mathfrak{u} \cap \mathfrak{p})$ and so on. We choose a system $\Delta^+(\mathfrak{g}, \mathfrak{t})$ of positive roots such that $\Delta(\mathfrak{u}) \subset \Delta^+(\mathfrak{g}, \mathfrak{t})$. As the Killing form is non-degenerate there exists for any $\alpha \in \Delta$ a vector $H_\alpha \in \mathfrak{t}$ so that for all $H \in \mathfrak{t}$, $\alpha(H) = B(H, H_\alpha)$.

Lemma 8. *There exists an orthonormal basis $(E_\alpha)_{\alpha\in\Delta}$ of root vectors satisfying*

$$[E_\alpha, E_{-\alpha}] = H_\alpha \tag{5a}$$
$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta} \quad \textit{with } N_{\alpha,\beta} = 0 \textit{ if } \alpha + \beta \notin \Delta \tag{5b}$$
$$N_{\alpha,\beta} = -N_{-\alpha,-\beta} \,. \tag{5c}$$

Proof. According to [Hel62, theorem 5.5] there exists a basis (E'_α) satisfying equations (5). The relation (5a) implies that $B(E'_\alpha, E'_{-\alpha}) = 1$. Moreover $B(E'_\alpha, E'_\beta) = 0$ if $\alpha + \beta \neq 0$, and $\|E'_\alpha\| > 0$, so it follows that $-\theta(\overline{E'_\alpha}) = c_{-\alpha} E'_{-\alpha}$, with $c_\alpha c_{-\alpha} = 1$. We now define $E_\alpha = x_\alpha E'_\alpha$ where $x_\alpha x_{-\alpha} = 1$ and $x_\alpha^2 = -c_\alpha$. We then get

$$-\theta(\overline{E_\alpha}) = x_\alpha c_{-\alpha} E'_{-\alpha} = -E_{-\alpha} \quad \text{and} \tag{6}$$
$$[E_\alpha, E_{-\alpha}] = x_\alpha x_{-\alpha} [E'_\alpha, E'_{-\alpha}] = H_\alpha \,. \tag{7}$$

So that $\langle E_\alpha, E_\alpha \rangle = B(E_\alpha, E_{-\alpha}) = 1$ and the basis (E_α) is now orthonormal. Using equations (6) and (7) is easy to check that the basis (E_α) again satifies the equations (5). □

We set $\overline{Z}_\alpha = E_\alpha$ and

$$Z_\alpha = \begin{cases} -E_{-\alpha} & \text{if } \alpha \text{ is compact,} \\ E_\alpha & \text{if } \alpha \text{ is non compact.} \end{cases}$$

This notation is concording with the complex structure. Let us now define the real vectors X_γ and Y_γ.

$$X_\gamma = \frac{1}{\sqrt{2}}\big(Z_\gamma + \overline{Z}_\gamma\big)\,, \quad Y_\gamma = -\frac{i}{\sqrt{2}}\big(\overline{Z}_\gamma - Z_\gamma\big)\,.$$

The sytem $(X_\gamma, Y_\gamma)_{\gamma\in\Delta^+\backslash Delta^+(\mathfrak{l}\cap\mathfrak{k})}$ is an orthonormal basis of T_eZ and $(X_\gamma, Y_\gamma)_{\gamma\in\Delta(\mathfrak{u}}$ is an orthonormal basis of $E_e \simeq T_eY$. Moreover if J denotes the complex multiplication operator, one has $Y_\gamma = JX_\gamma$, for $\gamma \in \Delta(\mathfrak{u})$ (and $\gamma \in \Delta(\mathfrak{l}\cap\mathfrak{p})$ when $L/L\cap K$ is a hermitian symertic space). We also have

$$X_\alpha = \frac{1}{\sqrt{2}}(E_\alpha - E_{-\alpha}) \quad Y_\alpha = -\frac{i}{\sqrt{2}}(E_\alpha + E_{-\alpha}) \qquad \text{if } \alpha \text{ is compact,} \tag{8a}$$

$$X_\beta = \frac{1}{\sqrt{2}}(E_\beta + E_{-\beta}) \quad Y_\beta = -\frac{i}{\sqrt{2}}(E_\beta - E_{-\beta}) \qquad \text{if } \beta \text{ is non compact.} \tag{8b}$$

Proposition 9. *For $\alpha \in \Delta(\mathfrak{u}\cap\mathfrak{k})$ and $\beta \in \Delta(\mathfrak{u}\cap\mathfrak{p})$ we have*

$$[X_\alpha, X_\beta] = \frac{1}{\sqrt{2}}\Big(N_{\alpha\beta}X_{\alpha+\beta} + N_{\alpha,-\beta}X_{|\alpha-\beta|}\Big) \tag{9a}$$

$$[X_\alpha, Y_\beta] = \frac{1}{\sqrt{2}}\Big(N_{\alpha,\beta}Y_{\alpha+\beta} - \epsilon(\alpha-\beta)N_{\alpha,-\beta}Y_{|\alpha-\beta|}\Big) \tag{9b}$$

$$[Y_\alpha, X_\beta] = \frac{1}{\sqrt{2}}\Big(N_{\alpha,\beta}Y_{\alpha+\beta} + \epsilon(\alpha-\beta)N_{\alpha,-\beta}Y_{|\alpha-\beta|}\Big) \tag{9c}$$

$$[Y_\alpha, Y_\beta] = -\frac{1}{\sqrt{2}}\Big(N_{\alpha,\beta}X_{\alpha+\beta} - N_{\alpha,-\beta}X_{|\alpha-\beta|}\Big) \tag{9d}$$

The vectors involving roots of the form $\alpha+\beta$ lie in E_e. The vectors involving roots of the form $\alpha-\beta$ may lie in F_e, but don't need to. Other brackets of base vectors lie in E_e.

To prove this proposition one just computes using equations (5b,5c) and the fact that if $\alpha \in \Delta(\mathfrak{u}\cap\mathfrak{k})$ is compact and $\beta \in \Delta(\mathfrak{u}\cap\mathfrak{p})$ is non compact then $\alpha\pm\beta$ either is a non compact root or is not a root.

Let e_γ be the exterior multiplication by Z_γ. Then the Dolbeault operator has the following principal E-symbol.

$$\overline{\partial} \simeq \sum_{\gamma\in\Delta(\mathfrak{u})} e_\gamma \overline{Z}_\gamma\,,$$

where $\overline{Z}_\gamma$ is here the left invariant vector field generated by $\overline{Z}_\gamma$. Let i_γ be the interior multiplication by $\overline{Z_\gamma}$ with respect to the chosen metric. Then

$$\overline{\partial}^* \simeq -\sum_{\gamma\in\Delta(\mathfrak{u})} i_\gamma Z_\gamma\,.$$

According to the previous notations these equations become

$$\overline{\partial} \simeq \sum_{\gamma\in\Delta(\mathfrak{u})} \frac{e_\gamma}{\sqrt{2}}\big(X_\gamma - iY_\gamma\big)\,,$$

$$\overline{\partial}^* \simeq -\sum_{\gamma\in\Delta(\mathfrak{u})} \frac{e_\gamma}{\sqrt{2}}\big(X_\gamma + iY_\gamma\big)\,.$$

It now remains to compute.

$$\overline{\square} \simeq -\frac{1}{2}\Big(\sum_{\gamma} e_\gamma \left(X_\gamma - iY_\gamma\right) \cdot \sum_{\gamma'} i_{\gamma'} \left(X_{\gamma'} + iY_{\gamma'}\right) + \sum_{\gamma'} i_{\gamma'} \left(X_{\gamma'} + iY_{\gamma'}\right) \cdot \sum_{\gamma} e_\gamma \left(X_\gamma - iY_\gamma\right)\Big) .$$

Let us write the diagonal terms separetely.

$$\begin{aligned}\overline{\square} \simeq & -\frac{1}{2} \sum_{\gamma\in\Delta(\mathfrak{u})} \left(e_\gamma i_\gamma + i_\gamma e_\gamma\right)\left(X_\gamma^2 + Y_\gamma^2\right) \\ & -\frac{1}{2}\sum_{\gamma\neq\gamma'} e_\gamma i_{\gamma'}\Big[\left(X_\gamma X_{\gamma'} + Y_\gamma Y_{\gamma'}\right) + i\left(X_\gamma Y_{\gamma'} + Y_\gamma X_{\gamma'}\right)\Big] \\ & \qquad + i_{\gamma'} e_\gamma \Big[\left(X_{\gamma'} X_\gamma + Y_{\gamma'} Y_\gamma\right) + i\left(Y_{\gamma'} X_\gamma + X_{\gamma'} Y_\gamma\right)\Big]\end{aligned}$$

We have $e_\gamma i_{\gamma'} + i_{\gamma'} e_\gamma = \delta_{\gamma\gamma'}$ (Kronecker' symbol).

$$\begin{aligned}\overline{\square} \simeq & -\frac{1}{2}\sum_{\gamma\in\Delta(\mathfrak{u})}\left(X_\gamma^2 + Y_\gamma^2\right) \\ & -\frac{1}{2}\sum_{\gamma\neq\gamma'} e_\gamma i_{\gamma'}\Big[\left([X_\gamma, X_{\gamma'}] + [Y_\gamma, Y_{\gamma'}]\right) + i\left([X_\gamma, Y_{\gamma'}] + [Y_\gamma, X_{\gamma'}]\right)\Big]\end{aligned}$$

Using proposition 9 one gets

$$[X_\gamma, X_{\gamma'}] + [Y_\gamma, Y_{\gamma'}] = \sqrt{2}\, N_{\alpha,-\beta} X_{|\alpha-\beta|} \quad \text{and}$$
$$[X_\gamma, Y_{\gamma'}] + [Y_\gamma, X_{\gamma'}] = -\sqrt{2}\, N_{\alpha,-\beta} Y_{|\alpha-\beta|} \,,$$

if $\gamma = \alpha$ is compact and $\gamma' = \beta$ is non compact. One has similar relations when $\gamma = \beta$ is non compact and $\gamma' = \alpha$ is compact. Other brackets are horizontal and they don't appear in the principal E-symbol. This gives

$$\begin{aligned}\overline{\square} \simeq -\frac{1}{2}\sum_{\gamma\in\Delta(\mathfrak{u})}\left(X_\gamma^2 + Y_\gamma^2\right) + \frac{\sqrt{2}}{2}\sum_{\gamma\in\Delta(\mathfrak{l}\cap\mathfrak{p})} & \Big[\Big({\textstyle\sum^*} N_{\alpha,\beta}\left(e_\alpha i_\beta - e_\beta i_\alpha\right)\Big) X_\gamma \\ & + i\Big({\textstyle\sum^*} N_{\alpha,\beta}\left(e_\alpha i_\beta + e_\beta i_\alpha\right)\Big) Y_\gamma\Big] \qquad (10)\end{aligned}$$

where the sums $\sum^*$ are over $\alpha \in \Delta(\mathfrak{u}\cap\mathfrak{k})$, $\beta \in \Delta(\mathfrak{u}\cap\mathfrak{p})$ and $|\alpha-\beta| = \gamma$. The local formula (10) will be used later in the proof of theorem 7.

This formula is already usefull for functions. In fact, the terms of classical order 1 vanish on functions, so $\overline{\square}$ is maximally hypoelliptic when restricted to functions because it has the same principal E-symbol as the Hörmander Laplacian (up to a constant).

2 The Rockland condition

2.1 Hypoellipticity criterion

For the proof of the theorem 7 we use techniques of Folland and Stein [FS74]. We now provide the tangent space T_eZ with a nilpotent Lie algebra structure $\mathfrak{n}_0$. This structure is given by the brackets $[\ ,\ ]_0$, and the identification of TZ/E with F. The Lie brackets $[\![\ ,\ ]\!]$ is then given as follows. Compare with proposition 9.

Definition 10. *For $\alpha \in \Delta(\mathfrak{u} \cap \mathfrak{k})$ and $\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p})$ we have*

$$[\![X_\alpha, X_\beta]\!] = \frac{1}{\sqrt{2}}\Big(N'_{\alpha,-\beta} X_{|\alpha-\beta|}\Big) \tag{11a}$$

$$[\![X_\alpha, Y_\beta]\!] = \frac{1}{\sqrt{2}}\Big(-\epsilon(\alpha-\beta) N'_{\alpha,-\beta} Y_{|\alpha-\beta|}\Big) \tag{11b}$$

$$[\![Y_\alpha, X_\beta]\!] = \frac{1}{\sqrt{2}}\epsilon(\alpha-\beta) N'_{\alpha,-\beta} Y_{|\alpha-\beta|}\Big) \tag{11c}$$

$$[\![Y_\alpha, Y_\beta]\!] = -\frac{1}{\sqrt{2}}\Big(-N'_{\alpha,-\beta} X_{|\alpha-\beta|}\Big), \tag{11d}$$

where $N'_{\alpha,-\beta} = N_{\alpha,-\beta}$ if $\alpha - \beta \in \Delta(\mathfrak{l} \cap \mathfrak{p})$ and 0 otherwise. All other brackets of base vectors are defined to be 0.

Let P be a differential operator on an open set of $\mathbb{R}^n$ as in the first part, with principal E-symbol p. We say that P satisfies the Rockland condition if for any unitary irreducible non trivial representation π of the simply connected nilpotent Lie group $N = \exp(\mathfrak{n})$, the operator $\pi(p)$ is injective on the space of smooth vectors of π. The sympbol p is seen here as an element of the enveloping algebra $U(\mathfrak{n})$ of $\mathfrak{n}$.

Theorem 11. *[HN85] The following are equivalent*

- *P has a parametrix in the E-pseudodifferential calculus*
- *P satifies to the Rockland condition*
- *P is maximal hypoellitptic*

Let N be a nilpotent Lie group with Lie algebra $\mathfrak{n}_0$. Then N acts on $\mathfrak{n}^*$ by the coadjoint representation. Kirillov defined a one-to-one correspondance between coadjoint orbits and (equivalence classes of) irreducible unitary representations of the group N constructed in three steps as follows.

Lemma 12. *Let l be a form on $\mathfrak{n}_0$ and $B_l\colon (X,Y) \mapsto l([X,Y])$. Then there exists an isotropic subalgebra $\mathfrak{h}_0$ of $\mathfrak{n}_0$ for B_l such that $\mathrm{codim}\mathfrak{h}_0 = \frac{1}{2}\mathrm{rank}B_l$.*

Then $\exp(il)$ is a one dimensionnal representation of the Nilpotent group H.

Lemma 13. *The induced representation $\mathrm{Ind}_H^N e^{il}$ is irreducible and its class only depends on the coadjoint orbit of l.*

There is also a converse statement.

Lemma 14. *All irreducible representations of N arises in this way exactly once.*

We will need to recognize induced representations realized on $\mathbb{R}^n$. Let π be a representation of the nilpotent Lie algebra $\mathfrak{n}_0$ on $\mathcal{S}(\mathbb{R}^n)$. We suppose that, for any $X \in \mathfrak{n}_0$, the operator $\pi(X)$ has the form

$$\pi(X) = \sum_{k=1}^{n-1} P_k(y_1, \ldots, y_{k-1}; X) \frac{\partial}{\partial y_k} + iQ(y_1, \ldots y_n; X)\,,$$

where $P_k(\cdot; X)$ and $Q(\cdot; X)$ are polynomials on $\mathbb{R}^n$ depending linearly on X. We also assume that the linear forms $\xi_k(X) = P_k(0; X)$ are linearly independent. Let l be the linear form on $\mathfrak{n}_0$ defined by $l(X) = Q(0; X)$ and $\mathfrak{h}_0 = \cap \ker \xi_k$.

Proposition 15. *[HN85, Proposition 1.6.1] Under the above assumptions, the subspace $\mathfrak{h}_0$ is a subalgebra of $\mathfrak{n}_0$, isotropic for B_l. Moreover, the representation π is unitarily equivalent to $\mathrm{Ind}_H^N e^{il}$.*

2.2 Proof of the main theorem

Here we prove that the evaluation of the Dolbeault laplacian has a kernel of positive dimension on many representations under conditions on root systems. The choice of these representations et the check of the root systems conditions are made for the groups $G = \mathrm{U}(p, q)$ and $L = \mathrm{U}(p_1) \times \mathrm{U}(p_2, q)$, with $p_1 + p_2 = p$. I expect that this can be done in full generality.

We now have to fine unitary irreducible representations of the connected nilpotent Lie group H and to realize them on $L^2(\mathbb{R}^n)$. This will lead to a partial differential equation on $\mathbb{R}^n$. In other words the linear form on $\mathfrak{n}_0$ that gives the representation of N, has to be taken such that the obtained partial differential equation can be solved and has a non zero solution space. Let $l \in \mathfrak{n}_0^*$ be a linear form on $\mathfrak{n}_0$ with coordinates $(\xi_\gamma, \eta_\gamma)$ in the dual basis of (X_γ, Y_γ). Let π_l be the representation of N associated to the coadjoint orbit of l.

Using definiton 10 one find that the form $B_l \colon (X, Y) \mapsto l([X, Y])$ as a martix of the form

$$\begin{pmatrix} 0 & A & 0 \\ -A^{\mathrm{t}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{with } A = \frac{N'_{\alpha,-\beta}}{\sqrt{2}} \begin{pmatrix} \xi_{|\alpha-\beta|} & -\varepsilon(\alpha-\beta)\eta_{|\alpha-\beta|} \\ \varepsilon(\alpha-\beta)\eta_{|\alpha-\beta|} & \xi_{|\alpha-\beta|} \end{pmatrix}_{\alpha,\beta}.$$

We make the following assumption on l.
(H) A has a maximal rank.
If hypothesis (H) is true then
(H') either $\mathfrak{p}_0$ or $\mathfrak{l}_0 \cap \mathfrak{p}_0 \oplus (\mathfrak{u} \oplus \overline{\mathfrak{u}}) \cap \mathfrak{k}_0$ is a maximal abelian subalgebra of $\mathfrak{n}_0$.
This means that the hypothesis (H) is more an hypothesis on the pair (G, Q) than on the linear form l. Let us assume hypothesis (H). Let $\mathfrak{h}_0$ be the abelian subalgebra of $\mathfrak{n}_0$ such that

$$\mathfrak{h}_0 = \mathfrak{p}_0 \quad \text{if} \quad \dim \mathfrak{p}_0 = \max\big\{ \dim \mathfrak{p}_0 \,;\, \dim \mathfrak{l}_0 \cap \mathfrak{p}_0 \oplus (\mathfrak{u} \oplus \overline{\mathfrak{u}}) \cap \mathfrak{k}_0 \big\}\,,$$

and $\mathfrak{h}_0 = \mathfrak{l}_0 \cap \mathfrak{p}_0 \oplus (\mathfrak{u} \oplus \overline{\mathfrak{u}}) \cap \mathfrak{k}_0$ otherwise. Then

$$\mathrm{codim}\mathfrak{h}_0 = \frac{1}{2}\mathrm{rank}B_l\,,$$

and $\mathfrak{h}_0$ is an isotropic subspace for B_l.

Lemma 16. *Let $G = \mathrm{U}(p,q)$ and $L = \mathrm{U}(p_1) \times \mathrm{U}(p_2,q)$. There exists a linear form l such that hypothesis (H) is satisfied.*

Proof. Take l be non zero on root vectors correszponding to a set of strongly orthogonal roots in $\Delta^+(\mathfrak{l} \cap \mathfrak{p})$ such as in the proof of lemma 4, and 0 elsewhere. Then A is "diagonal" with no zero on the diagonal, by lemmas 3,4. □

First case. Let us begin with the case $\mathfrak{h}_0 = \mathfrak{p}_0$. Let $s = \dim_{\mathbb{C}} K/L \cap K = \dim \mathfrak{u} \cap \mathfrak{k}$. Then $\pi_l = Ind_H^N e^{il}$ is a unitary irreducible representation of N on $L^2(\mathfrak{n}_0/\mathfrak{h}_0)$ that can be seen as a representation on $L^2(\mathbb{R}^{2s})$. We note $(x_\alpha, y_\alpha)_{\alpha\in\Delta(\mathfrak{u}\cap\mathfrak{k})}$ the canonical basis of $\mathbb{R}^{2s}$. Thanks to proposition 15, we have

$$\begin{aligned}
\pi_l(X_\alpha) = \frac{\partial}{\partial x_\alpha} + i\xi_\alpha \quad \pi_l(Y_\alpha) = \frac{\partial}{\partial y_\alpha} + i\eta_\alpha \\
\pi_l(X_\beta) = i\sum_\alpha \left[\frac{N'_{\alpha,-\beta}}{\sqrt{2}}(\xi_{|\alpha-\beta|}x_\alpha - \varepsilon(\alpha-\beta)\eta_{|\alpha-\beta|}y_\alpha)\right] + i\xi_\beta \\
\pi_l(Y_\beta) = i\sum_\alpha \left[\frac{N'_{\alpha,-\beta}}{\sqrt{2}}(\varepsilon(\alpha-\beta)\eta_{|\alpha-\beta|}x_\alpha + \xi_{|\alpha-\beta|}y_\alpha)\right] + i\eta_\beta \\
\pi_l(X_\gamma) = i\xi_\gamma \quad \pi_l(Y_\gamma) = i\eta_\gamma
\end{aligned}$$

To make the computation more easy we also suppose that

$$\xi_\alpha = \eta_\alpha = \xi_\beta = \eta_\beta = 0\,.$$

This is not true in general that any orbits admits a form of this kind, but this is enough, to prove the theorem, to find such forms such that $\pi_l(\overline{\Box})$ is not injective. Then, the operator $\pi_l\left(\overline{\Box}\right)$ has the following form.

$$\pi_l\left(\overline{\Box}\right) = -\frac{1}{2}\sum_\alpha \left[\frac{\partial^2}{\partial x_\alpha^2} + \frac{\partial^2}{\partial y_\alpha^2} - r_\alpha^2(x_\alpha^2 + y_\alpha^2)\right] + \sum_\alpha \left[{\sum}^* M_{\alpha,\beta}\right]\,, \tag{12}$$

where r_α is the positive real number such that $r_\alpha^2 = \sum^* \frac{{N'_{\alpha,-\beta}}^2}{2}(\xi_{|\alpha-\beta|}^2 + \eta_{|\alpha-\beta|}^2)$, and

$$M_{\alpha,\beta} = \frac{iN'_{\alpha,-\beta}}{\sqrt{2}}\left[(\xi_{|\alpha-\beta|} + i\eta_{|\alpha-\beta|})e_\alpha i_\beta - (\xi_{|\alpha-\beta|} - i\eta_{|\alpha-\beta|})i_\beta e_\alpha\right]$$

is an endomorphism of $\wedge^*\mathfrak{u}$ and the sum $\sum^*$ is over the set of roots $\beta \in \Delta(\mathfrak{u}\cap\mathfrak{p})$ such that $\alpha - \beta \in \Delta(\mathfrak{l}\cap\mathfrak{p})$.

Let $D_\alpha = -\frac{1}{2}\left[\frac{\partial^2}{\partial x_\alpha^2} + \frac{\partial^2}{\partial y_\alpha^2} - r_\alpha^2(x_\alpha^2 + y_\alpha^2)\right]$ and $M_\alpha = \sum^* M_{\alpha,\beta}$. We have to find eigenvalues of $\sum_\alpha D_\alpha$ and $\sum_\alpha M_\alpha$ of opposite signs and the same absolute value. Making the change of variables

$$x_\alpha \mapsto r_\alpha^{\frac{1}{2}} x_\alpha \quad y_\alpha \mapsto r_\alpha^{\frac{1}{2}} y_\alpha \,,$$

the operator D_α becomes $-\frac{r_\alpha}{2}\left[\frac{\partial^2}{\partial x_\alpha^2} + \frac{\partial^2}{\partial y_\alpha^2} - (x_\alpha^2 + y_\alpha^2)\right]$. It is $-\frac{r_\alpha}{2}$ times the Hermite operator of dimension 2. Its eigenvalues are then $-kr_\alpha$, with $k \in \mathbb{N}^*$. As the operators D_α differentiate on different variables, we see that the eigenvalues of $\sum_\alpha D_\alpha$ are $-\sum_\alpha k_\alpha r_\alpha$, with $k_\alpha \in \mathbb{N}^*$. We also note that the eigenfunctions of the Hermite operator are of the form $Pe^{-\frac{x^2+y^2}{2}}$ where P is a polynomial in x and y. So they are in the Schwarz space, so are smooth vectors of the representation π_l.

Let us now show that $\pm \sum_\alpha r_\alpha$ is an eigenvalue of $\sum_\alpha M_\alpha$. We first show that r_α is an eigenvalue of M_α. Let $\Delta(\mathfrak{u} \cap \mathfrak{k}) = \{\alpha_1, \ldots, \alpha_s\}$ and $v = Z_{\alpha_1} \wedge \cdots \wedge Z_{\alpha_s}$. If $\beta \neq \beta'$, then $M_{\alpha,\beta} M_{\alpha,\beta'}(v) = M_{\alpha,\beta'} M_{\alpha,\beta}(v) = 0$ and moreover

$$M_{\alpha,\beta}^2(v) = \frac{N_{\alpha,-\beta}'^2}{2}(\xi_{|\alpha-\beta|}^2 + \eta_{|\alpha-\beta|}^2)\,.$$

It follows that

$$M_\alpha^2(v) = \sum{}^* M_{\alpha,\beta}^2(v) = r_\alpha^2 v\,.$$

So the vector $\pm r_\alpha v + M_\alpha v$ is an eigenvector for M_α with eigenvalue $\pm r_\alpha$.

Proposition 17. *Let $k \leq s$ and $\{i_1; \cdots ; i_k\} \subset \{1; \cdots ; s\}$. Then $\prod_{l=1}^k M_{\alpha_{i_l}} v$ does not depend on the order of the i_l.*

This proposition is easily checked by induction on k. We now define by induction, for $k \leq s$, the vectors v_k by $v_0 = v$ and

$$v_k = (r_{\alpha_k} + M_{\alpha_k}) v_{k-1}\,.$$

The preceding proposition shows that if v_{k-1} is an eigenvector for M_{α_l}, $l < k$, with eigenvalue r_{α_l}, then v_k is again an eigenvector for M_{α_l}, $l < k$, with eigenvalue r_{α_l}.

Lemma 18. *Let $G = \mathrm{U}(p,q)$ and $L = \mathrm{U}(p_1) \times \mathrm{U}(p_2, q)$. There exists a linear form l on $\mathfrak{n}_0$ satisfying hypothesis (H), and such that v_k is an eigenvector for M_{α_k}, with eigenvalue r_{α_k}.*

Proof. Take l has in the proof of lemma 16 again works. □

Hence v_s is a simultaneous eigenvector for all M_α's, with respective eigenvalue r_α. So v_s is an eigenvector for $\sum_\alpha M_\alpha$ with eigenvalue $\sum_\alpha r_\alpha$. We end this first case $\mathfrak{h} = \mathfrak{p}_0$ remarking that the constructed eigenvector lies in $\wedge^s \mathfrak{u}$, and this means that $\overline{\Box}$ is not maximally hypoelliptic on degree $s = \dim_{\mathbb{C}} K/L \cap K$.

Second case. Let us now assume that $t = \dim \mathfrak{u} \cap \mathfrak{p} < s$. Switching the role played in the first case by the α's and the β's, one similiraly prove that $\overline{\Box}$ is not maximally hypoelliptic on $\wedge^t \mathfrak{u}$. Using the duality

$$\wedge \colon \wedge^t \mathfrak{u} \otimes \wedge^s \mathfrak{u} \to \wedge^{\max} \mathfrak{u}\,,$$

one shows that $\overline{\Box}$ is not maximally hypoelliptic on degree s in the second case too.

Finally, we have shown that $\overline{\Box}$ is never hypoellitpic on degree s and on the complementary degree t.

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