

EXHAUSTING FAMILIES OF REPRESENTATIONS AND SPECTRA OF PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. A powerful tool in the spectral theory and the study of Fredholm conditions for (pseudo)differential operators is provided by families of representations of a naturally associated algebra of bounded operators. Motivated by this approach, we define the concept of an *exhausting family* of representations of a C^* -algebra A . Let \mathcal{F} be an exhausting family of representations of A . We have then that an abstract differential operator D affiliated to A is invertible if, and only if, $\phi(D)$ is invertible for all $\phi \in \mathcal{F}$. This property characterizes exhausting families of representations. We provide necessary and sufficient conditions for a family of representations to be exhausting. If A is a separable C^* -algebra, we show that a family \mathcal{F} of representations is exhausting if, and only if, every irreducible representation of A is (weakly) contained in a representation $\phi \in \mathcal{F}$. However, this result is not true, in general, for non-separable C^* -algebras. A typical application of our results is to parametric families of differential operators arising in the analysis on manifolds with corners, in which case we recover the fact that a parametric operator P is invertible if, and only if, its Mellin transform $\hat{P}(\tau)$ is invertible, for all $\tau \in \mathbb{R}^n$. The paper is written to be accessible to non-specialists in C^* -algebras.

CONTENTS

INTRODUCTION

Let us begin by motivating the present work using spectral theory and the related Fredholm conditions for pseudodifferential operators. A typical result in spectral theory of N -body Hamiltonians [?, ?, ?, ?] associates to the Laplacian H a family of other operators H_ϕ , $\phi \in \mathcal{F}$, such that the essential spectrum $\text{Spec}_{ess}(H)$ of H is obtained in terms of the usual spectra $\text{Spec}(H_\phi)$ of H_ϕ as the closure of the union of the later:

$$(1) \quad \text{Spec}_{ess}(H) = \overline{\bigcup_{\phi \in \mathcal{F}} \text{Spec}(H_\phi)}.$$

It was noticed that sometimes the closure is not necessary, and one of the motivations of our paper is to clarify this issue. It is well known that the operators H_ϕ are obtained as homomorphic images of the operator H , that is $H_\phi = \phi(H)$, where the morphisms $\phi \in \mathcal{F}$ are defined on a certain C^* -algebra associated to H . This justifies the study of families of representations. See for example [?] for results in this direction.

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Another motivation comes from the characterization of Fredholm operators (Fredholm conditions) for (pseudo)differential operators [?]. More precisely, for suitable manifolds M and for differential operators D on M compatible with the geometry, there was devised a procedure to associate to M the following data:

- (i) spaces Z_α , $\alpha \in I$;
- (ii) groups G_α , $\alpha \in I$; and
- (iii) G_α -invariant differential operators D_α acting on $Z_\alpha \times G_\alpha$.

This data can be used to characterize the Fredholm property of D as follows. Let m be the order of D , then

- (2) $D : H^s(M) \rightarrow H^{s-m}(M)$ is Fredholm \Leftrightarrow D is elliptic and
 D_α is invertible for all $\alpha \in I$.

Moreover, the spaces Z_α and the groups G_α are independent of D . If M is compact (without boundary), then the index I is empty (so there are no D_α s). In general, for non-compact manifolds, the conditions on the operators D_α are, however, necessary. The non-compact geometries to which this characterization of Fredholm operators applies include: asymptotically euclidean manifolds, asymptotically hyperbolic manifolds, manifolds with poly-cylindrical ends, and many others. Again, the operators P_α are homomorphic images of the operator P , which motivates the study of families of representations.

The results in [?, ?, ?] mentioned above are the main motivation for this work, which is a purely theoretical one on the representation theory of C^* -algebras, even though the applications are to spectral theory and (pseudo)differential operators. We thus define the concept of an *exhausting family* \mathcal{F} of representations of a C^* -algebra A as having the property that for any $a \in A$, there exists $\phi \in \mathcal{F}$ such that $\|a\| = \|\phi(a)\|$. We have learned from G. Skandalis that he has also considered this condition (private communication). The family \mathcal{F} does not have to consist of irreducible representations. Let \mathcal{F} be an exhausting family of representations of A , we show then that an abstract differential operator D affiliated to A is invertible if, and only if, $\phi(D)$ is invertible for all $\phi \in \mathcal{F}$. This property characterizes exhausting families of representations. We provide a necessary and sufficient conditions for a family of representations to be exhausting in terms of the topology on the Jacobson primitive ideal spectrum \hat{A} of A . If A is a separable C^* -algebras, we show that a family \mathcal{F} of representations is exhausting if, and only if, every irreducible representation of A is (weakly) contained in a representation $\phi \in \mathcal{F}$.

A typical application is to parametric families of differential operators arising in the analysis on manifolds with corners (more precisely, in the case of manifolds with polycylindrical ends). In that case, we recover the fact that an operator compatible with the geometry is invertible if, and only if, its Mellin transform is invertible. We discuss also several other examples. Due to the nature of the main applications to other areas than the study of C^* -algebras, we write the paper with an eye towards the non-specialist in C^* -algebras.

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1. C^* -ALGEBRAS AND THEIR PRIMITIVE IDEAL SPECTRUM

We begin with a review of some needed general C^* -algebra results. We recall [?] that a C^* -algebra is a complex algebra A together with a conjugate linear involution $*$ and a complete norm $\| \cdot \|$ such that $(ab)^* = b^*a^*$, $\|ab\| \leq \|a\|\|b\|$, and $\|a^*a\| = \|a\|^2$, for all $a, b \in A$. (The fact that $*$ is an involution means that $a^{**} = a$.) In particular, a C^* -algebra is also a Banach algebra. Let \mathcal{H} be a Hilbert space and denote by $\mathcal{L}(\mathcal{H})$ the space of linear, bounded operators on \mathcal{H} . One of the main reasons why C^* -algebras are important is that every norm-closed subalgebra $A \subset \mathcal{L}(\mathcal{H})$ that is also closed under taking Hilbert space adjoints is a C^* -algebra. Abstract C^* -algebras have many non-trivial properties that can then be used to study the concretely given algebra A . Conversely, every abstract C^* -algebra is isometrically isomorphic to a norm closed subalgebra of $\mathcal{L}(\mathcal{H})$ (the Gelfand-Naimark theorem, see [?, theorem 2.6.1]).

Let A denote a generic C^* -algebra throughout this paper. A *representation* of A on the Hilbert space \mathcal{H}_π is a $*$ -morphism $\pi : A \rightarrow \mathcal{L}(\mathcal{H}_\pi)$ to the algebra of bounded operators on \mathcal{H}_π . We shall use the fact that every morphism ϕ of C^* -algebras (and hence any representation of a C^* -algebra) has norm $\|\phi\| \leq 1$. Consequently, every bijective morphism of C^* -algebras is an isometric isomorphism, and, in particular

$$(3) \quad \|\phi(a)\| = \|a + \ker(\phi)\|_{A/\ker(\phi)}.$$

A two-sided ideal $I \subset A$ is called *primitive* if it is the kernel of an irreducible representation. We shall denote by \hat{A} the set of primitive ideals of A . For any two-sided ideal $J \subset A$, we have that its primitive ideal spectrum \hat{J} identifies with the set of all the primitive ideals of A *not* containing the two-sided ideal $J \subset A$. It turns out then that the sets of the form \hat{J} , where J ranges through the set of two-sided ideals $J \subset A$, define a topology on \hat{A} , called the *Jacobson topology* on \hat{A} .

By $\phi : A \rightarrow \mathcal{L}(\mathcal{H}_\phi)$ we shall denote generic representations of A . For any representation ϕ of A , we define its support, $\text{supp}(\phi) \subset \hat{A}$ as the complement of $\widehat{\ker(\phi)}$, that is, $\text{supp}(\phi) := \hat{A} \setminus \widehat{\ker(\phi)}$ is the set of primitive ideals of A *containing* $\ker(\phi)$.

Remark 1.1. The irreducible representations of A do not form a set (there are too many of them). The *unitary equivalence classes* of irreducible representations of A do form a set however, which we shall denote by $\text{Irr}(A)$. By $\pi : A \rightarrow \mathcal{L}(\mathcal{H}_\pi)$ we shall denote an arbitrary *irreducible* representation of A . There exists then by definition a surjective map

$$(4) \quad \text{can} : \text{Irr}(A) \rightarrow \hat{A}$$

that associates to (the class of) each irreducible representation $\pi \in \text{Irr}(A)$ its kernel $\ker(\pi)$. For each $a \in A$ and each irreducible representation π of A , the algebraic properties of $\pi(a)$ depend only on the kernel of π . That yields a well defined function

$$(5) \quad \text{can} : \text{Irr}(A) \ni \pi \rightarrow \|\pi(a)\| \in [0, \|a\|],$$

which descends to a well defined function

$$(6) \quad n_a : \hat{A} \ni \pi \rightarrow \|\pi(a)\| \in [0, \|a\|], \quad n_a(\ker(\pi)) = \|\pi(a)\|.$$

A C^* -algebra is called *type I* if, and only if, the surjection $\text{can} : \text{Irr}(A) \ni \pi \rightarrow \hat{A}$ of Equation (??) is, in fact, a bijection. Then the discussion of Remark ?? becomes

unnecessary and several arguments below will be (slightly) simplified since we will not have to make distinction between equivalence classes of irreducible representations and their kernels. Fortunately, many (if not all) of the C^* -algebras that arise in the study of pseudodifferential operators and of other practical questions are type I C^* -algebras. In spite of this, it seems unnatural at this time to restrict our study to type I C^* -algebras. Therefore, we will not assume that A is a type I C^* -algebra, unless this assumption is really needed.

We shall need the following simple (and well known) lemma.

Lemma 1.2. *The map $n_a : \hat{A} \ni \pi \rightarrow \|\pi(a)\| \in [0, \|a\|]$ is lower semi-continuous, that is, the set $\{\pi \in \hat{A}, \|\pi(a)\| > t\}$ is open for any $t \in \mathbb{R}$.*

We include the simple proof for the benefit of the non-specialist.

Proof. Let us fix $t \in \mathbb{R}$. Since n_a takes on non-negative values, we may assume $t \geq 0$. Let then $\chi : [0, \infty) \rightarrow [0, 1]$ be a continuous function that is zero on $[0, t^2]$ but is > 0 on (t^2, ∞) and let $b = \chi(a^*a)$, which is defined using the functional calculus with continuous functions. If $\phi : A \rightarrow \mathcal{L}(\mathcal{H}_\phi)$ is a representation of A , then we have that $\|\phi(a)\|^2 = \|\phi(a^*a)\| \leq t^2$ if, and only if,

$$\chi(\phi(a^*a)) = \phi(\chi(a^*a)) = \phi(b) = 0.$$

Let then J be the (closed) two sided ideal generated by b , that is, $J := \overline{AbA}$. Then

$$\{\pi \in \hat{A}, \|\pi(a)\| \leq t\} = \{\pi \in \hat{A}, \pi(b) = 0\} = \{\pi \in \hat{A}, \pi(J) = 0\} = \hat{A} \setminus \hat{J},$$

is hence a closed set. Consequently, $\{\pi \in \hat{A}, \|\pi(a)\| > t\}$ is open, as claimed. \square

2. FAITHFUL FAMILIES

Let \mathcal{F} be a set of representations of A . We say that the family \mathcal{F} is *faithful* if the direct sum representation $\rho := \bigoplus_{\phi \in \mathcal{F}} \phi$ is injective. We have the following well known result that will serve us as a model for characterization of “exhausting families” of representations in the next subsection.

Proposition 2.1. *Let \mathcal{F} be a family of representations of the C^* -algebra A . The following are equivalent:*

- (i) *The family \mathcal{F} is faithful.*
- (ii) *The union $\bigcup_{\phi \in \mathcal{F}} \text{supp}(\phi)$ is dense in \hat{A} .*
- (iii) *$\|a\| = \sup_{\phi \in \mathcal{F}} \|\phi(a)\|$ for all $a \in A$.*

Proof. (i) \Rightarrow (ii). We proceed by contradiction. Let us assume (i), but that (ii) is not true. That is, we assume that $\bigcup_{\phi \in \mathcal{F}} \text{supp}(\phi)$ is not dense in \hat{A} . Then there exists a non empty open set $\hat{J} \subset \hat{A}$ that does not intersect $\bigcup_{\phi \in \mathcal{F}} \text{supp}(\phi)$, where $J \subset A$ is a non-trivial two-sided ideal. Then $J \neq 0$ is contained in the kernel of $\bigoplus_{\phi \in \mathcal{F}} \phi$ and hence \mathcal{F} is not faithful. This is a contradiction, and hence (ii) must be true if (i) is true.

(ii) \Rightarrow (iii). For a given $a \in A$, the map sending the kernel $\ker \pi$ of an irreducible representation π to $\|\pi(a)\|$ is a lower semi-continuous function $\hat{A} \rightarrow [0, \infty)$, by Lemma ???. Moreover, for any $a \in A$ there exists an irreducible representation π_a such that $\|\pi_a(a)\| = \|a\|$. Hence, for every $\epsilon > 0$, $\{\pi \in \hat{A}, \|\pi(a)\| > \|a\| - \epsilon\}$ is a non empty open set (it contains $\ker \pi_a$) and then it contains some $\pi \in \bigcup_{\phi \in \mathcal{F}} \text{supp}(\phi)$,

since the later set was assumed to be dense in \hat{A} . Let $\phi \in \mathcal{F}$ be such that $\ker(\pi) \supset \ker(\phi)$. Then

$$\|a\| \geq \|\phi(a)\| \geq \|\pi(a)\| > \|a\| - \varepsilon,$$

where the first inequality is due to the general fact that representations of C^* -algebras have norm ≤ 1 and the second one is due to the fact that

$$\|\phi(a)\| = \|a + \ker(\phi)\|_{A/\ker(\phi)} \geq \|a + \ker(\pi)\|_{A/\ker(\pi)} = \|\pi(a)\|,$$

by Equation (??). Consequently, $\|a\| = \sup_{\phi \in \mathcal{F}} \|\phi(a)\|$, as desired.

(iii) \Rightarrow (i). Let $\rho := \oplus_{\phi \in \mathcal{F}} \phi : A \rightarrow \oplus_{\phi \in \mathcal{F}} \mathcal{L}(H_\phi)$. We need to show that ρ is injective. The norm on $\oplus_{\phi \in \mathcal{F}} \mathcal{L}(H_\phi)$ is the sup norm, that is, $\|(T_\phi)_{\phi \in \mathcal{F}}\| = \sup_{\phi \in \mathcal{F}} \|T_\phi\|$. Therefore $\|\rho(a)\| = \sup_{\phi \in \mathcal{F}} \|\phi(a)\| = \|a\|$, since we are assuming (iii). Consequently, ρ is isometric, and hence it is injective. \square

In the next proposition we shall need to assume that A is unital (that is, that it has a unit $1 \in A$). This assumption is not very restrictive since, given any non-unital C^* -algebra A_0 , the algebra with adjoint unit $A = A_0^+ := A_0 \oplus \mathbb{C}$ has a unique C^* -algebra norm.

For any unital C^* -algebra A and any $a \in A$, we denote by $\text{Spec}_A(a)$ the *spectrum* of a in A , defined by

$$\text{Spec}_A(a) := \{ \lambda \in \mathbb{C}, \lambda - a \text{ is not invertible in } A \}.$$

It is known classically that $\text{Spec}_A(a)$ is compact and non-empty, unlike in the case of unbounded operators [?]. We shall need the following general property of C^* -algebras.

Lemma 2.2. *Let $A_1 \subset B$ be two C^* -algebras and $a \in A_1$ be such that it has an inverse in B , denoted a^{-1} . Then $a^{-1} \in A_1$. In particular, the spectrum of a is independent of the C^* -algebra in which we compute it:*

$$(7) \quad \text{Spec}_{A_1}(a) = \text{Spec}_B(a) =: \text{Spec}(a).$$

If $a \in A$ for some non-unital C^* -algebra, then we define $\text{Spec}(a) := \text{Spec}_{A^+}(a)$, where $A^+ := A \oplus \mathbb{C}$, so $\text{Spec}(a)$ is independent of the C^* -algebra containing a also in the non-unital case.

Proposition 2.3. *Let \mathcal{F} be a faithful family of representations of a unital C^* -algebra A . An element $a \in A$ is invertible if, and only if, $\phi(a)$ is invertible in $\mathcal{L}(H_\phi)$ for all $\phi \in \mathcal{F}$ and the set $\{\|\phi(a)^{-1}\|, \phi \in \mathcal{F}\}$ is bounded.*

Proof. If a is invertible, $\phi(a)$ also is invertible and $\|\phi(a)^{-1}\| = \|\phi(a^{-1})\| \leq \|a^{-1}\|$ is hence bounded.

Conversely, let ρ be the direct sum of all the representations $\phi \in \mathcal{F}$, that is,

$$(8) \quad \rho := \oplus_{\phi \in \mathcal{F}} \phi : A \longrightarrow \oplus_{\phi \in \mathcal{F}} \mathcal{L}(H_\phi).$$

If $\|\phi(a)\|$ is invertible for all $\phi \in \mathcal{F}$ and there exists M independent of ϕ such that $\|\phi(a)^{-1}\| \leq M$, then $b := (\phi(a)^{-1})_{\phi \in \mathcal{F}}$ is a well defined element in $B := \oplus_{\phi \in \mathcal{F}} \mathcal{L}(H_\phi)$ and b is an inverse for $\rho(a)$ in B . Let $A_1 := \rho(A)$. Then $\rho(a) \in A_1$ is invertible in B . By the Lemma ??, $\rho(a)$ is invertible in A_1 . Then observe that since ρ is continuous, injective, and surjective morphism of C^* -algebras, it defines an isomorphism of algebras $A \rightarrow A_1$. We then conclude that a is invertible in A as well. \square

The following is a converse of the above proposition. Recall that $a \in A$ is called *normal* if $aa^* = a^*a$.

Proposition 2.4. *Let \mathcal{F} be a family of representations of a unital C^* -algebra A with the following property:*

If $a \in A$ is such that $\phi(a)$ is invertible in $\mathcal{L}(\mathcal{H}_\phi)$ for all $\phi \in \mathcal{F}$ and the set $\{\|\phi(a)^{-1}\|, \phi \in \mathcal{F}\}$ is bounded, then a is invertible in A .

Then the family \mathcal{F} is faithful.

Proof. Clearly, the family \mathcal{F} is not empty, since otherwise all elements of A would be invertible, which is not possible. Let us assume, by contradiction, that the family \mathcal{F} is *not* faithful. Then, by Proposition ??(ii), there exists a non-empty open set $V \subset \hat{A}$ that does not intersect $\cup_{\phi \in \mathcal{F}} \text{supp}(\phi)$. Let $J \subset A$, $J \neq 0$, be the (closed) two-sided ideal corresponding to V , that is, $V = \hat{J}$. Since \mathcal{F} is non-empty, we have $J \neq \hat{A}$. Then every $\phi \in \mathcal{F}$ is such that $\phi = 0$ on J . Let $a \in J$, $a \neq 0$. By replacing a with $a^*a \in J$, we can assume $a \geq 0$. Let $\lambda \in \text{Spec}(a)$, $\lambda \neq 0$. Such a λ exists since $a \neq 0$ and is normal and $a \neq 0$. Let $c := \lambda - a$. Then, for any $\phi \in \mathcal{F}$, $\phi(c) = \lambda \in \mathbb{C}$ is invertible and $\|\phi(c)^{-1}\| = \lambda^{-1}$ is bounded. However, c is not invertible (in any C^* -algebra containing it) since it belongs to the non-trivial ideal J . \square

Recall that $\mathcal{C}_0(X)$ is the set of continuous functions on X that have vanishing limit at infinity. Then $\mathcal{C}_0(X)$ is a commutative C^* -algebra, and all commutative C^* -algebras are of this form.

Example 2.5. Let μ_α , $\alpha \in I$, be a family of positive, regular Borel measures on a locally compact space X . Let ϕ_α be the corresponding multiplication representation of the C^* -algebra $\mathcal{C}_0(X) \rightarrow \mathcal{L}(L^2(X, \mu_\alpha))$. We have $\text{supp}(\phi_\alpha) = \text{supp}(\mu_\alpha)$ and the family $\mathcal{F} := \{\phi_\alpha, \alpha \in I\}$ is faithful if, and only if, $\cup_{\alpha \in I} \text{supp}(\mu_\alpha)$ is dense in X . In particular, if each μ_α is the Dirac measure concentrated at some $x_\alpha \in X$, then $\phi_\alpha(f) = f(x_\alpha) =: \text{ev}_{x_\alpha}(f) \in \mathbb{C}$ and $\text{supp}(\mu_\alpha) = \{x_\alpha\}$. We shall henceforth identify $x_\alpha \in X$ with the corresponding evaluation irreducible representation ev_{x_α} . Then we have that

$$\mathcal{F} = \{\text{ev}_{x_\alpha}, \alpha \in I\} \text{ is faithful} \Leftrightarrow \{x_\alpha, \alpha \in I\} \text{ is dense in } X.$$

This example extends right away to C^* -algebras of the form $\mathcal{C}_0(X; \mathcal{K})$ of functions with values compact operators on some given Hilbert space.

We conclude our discussion of faithful families with the following result. We denote by $\overline{\cup S_\alpha} := \overline{\cup_\alpha S_\alpha}$ the closure of the union of the family of sets S_α .

Proposition 2.6. *Let \mathcal{F} be a family of representations of a unital C^* -algebra A . Then \mathcal{F} is faithful if, and only if,*

$$(9) \quad \text{Spec}(a) = \overline{\cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))}.$$

for any normal $a \in A$.

Proof. Let us assume first that the family \mathcal{F} is faithful and that a is normal. Since we have that $\text{Spec}(\phi_0(a)) \subset \text{Spec}(a)$ for any representation ϕ_0 of A , it is enough to show that $\text{Spec}(a) \subset \overline{\cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))}$. Let us assume the contrary and let $\lambda \in \text{Spec}(a) \setminus \overline{\cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))}$. By replacing a with $a - \lambda$, we can assume that $\lambda = 0$. We thus have that $\phi(a)$ is invertible for all $\phi \in \mathcal{F}$, but a is not invertible (in A). Moreover, $\|\phi(a)^{-1}\| \leq \delta^{-1}$, where δ is the distance from $\lambda = 0$ to the spectrum

of $\phi(a)$, by the properties of the functional calculus for normal operators. This is however a contradiction by Proposition ??, which implies that a must be invertible in A as well.

To prove the converse, let us assume that $\text{Spec}(a) \subset \overline{\cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))}$, for all normal elements $a \in A$. Let J be a non-trivial (closed *selfadjoint*) two-sided ideal on which all the representations $\phi \in \mathcal{F}$ vanish. We have to show that $J = 0$, which would imply that \mathcal{F} is faithful. Let $a \in J$ be a normal element. Then $\text{Spec}(a) \subset \overline{\cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))} = \{0\}$. Since a is normal we deduce $a = 0$ and hence J has no normal element other than 0. Then, for any $a \in J$, we can write $a = 1/2(a+a^*) + 1/2(a-a^*)$, the sum of two normal elements in J because J is selfadjoint. Therefore $1/2(a+a^*) = 1/2(a-a^*) = 0$, and hence $a = 0$ and $J = 0$. \square

3. FULL AND EXHAUSTING FAMILIES

Let us notice that Example ?? shows the ‘sup’ in the relation $\|a\| = \sup_{\phi \in \mathcal{F}} \|\phi(a)\|$ (Proposition ??) may not be attained. It also shows that the closure of the union in Equation (??) is needed. Sometimes, in applications, one does obtain however the stronger version of these results (that is, the sup is attained and the closure is not needed), see [?, ?], for example. Moreover, the condition that the norms of $\phi(a)^{-1}$ be uniformly bounded (in ϕ) for any fixed $a \in A$ is inconvenient and often not needed in applications. For this reason, we introduce now a more restrictive class of families of representations of A .

Recall that $\text{supp}(\phi)$ is the set of primitive ideals of A that contain $\ker(\phi)$.

Definition 3.1. Let \mathcal{F} be a set of representations of the C^* -algebra A .

- (i) We shall say that \mathcal{F} is *full* if $\hat{A} = \cup_{\phi \in \mathcal{F}} \text{supp}(\phi)$.
- (ii) We shall say that \mathcal{F} is *exhausting* if, for any $a \in A$, there exists $\phi \in \mathcal{F}$ such that $\|\phi(a)\| = \|a\|$.

We see that a family \mathcal{F} is exhausting if, and only if, for any $a \in A$,

$$\|a\| = \max_{\phi \in \mathcal{F}} \|\phi(a)\|.$$

Example 3.2. By classical results, the set of all irreducible representations of a C^* -algebra is exhausting. See also Theorem ??.

Let us record the following simple facts, for further use.

Proposition 3.3. *Let \mathcal{F} be a set of representations of the C^* -algebra A . If \mathcal{F} is full, then \mathcal{F} is exhausting. If \mathcal{F} is exhausting, then it is also faithful.*

Proof. Let us prove first that any full family \mathcal{F} is exhausting. Indeed, let $a \in A$. Then there exists an irreducible representation π of A such that $\|\pi(a)\| = \|a\|$. Let $\phi \in \mathcal{F}$ be such that $\pi \in \text{supp}(\phi)$, then, as in the proof of (ii) \Rightarrow (iii) in Proposition ??, we have that $\|a\| = \|\pi(a)\| \leq \|\phi(a)\| \leq \|a\|$. Hence $\|\phi(a)\| = \|a\|$.

Let us prove first that any exhausting family \mathcal{F} is faithful. Indeed, let us consider the representation $\rho := \oplus_{\phi \in \mathcal{F}} \phi : A \rightarrow \oplus_{\phi \in \mathcal{F}} \mathcal{L}(H_\phi)$. By the definition of an exhausting family of representations, the representation ρ is isometric. Therefore it is injective and consequently \mathcal{F} is faithful. \square

We summarize the above Proposition in

$$\mathcal{F} \text{ full} \Rightarrow \mathcal{F} \text{ exhausting} \Rightarrow \mathcal{F} \text{ faithful.}$$

In the next two examples we will see that there exist faithful families that are not exhausting and exhausting families that are not full.

Example 3.4. We consider again the framework of Example ?? and consider only families of *irreducible representations*. Thus $A = \mathcal{C}_0(X)$, for a locally compact space X . The irreducible representations of A then identify with the points of X , since $X \simeq \hat{A}$. A family \mathcal{F} of *irreducible* representations of A thus identifies with a subset $\mathcal{F} \subset X$. We then have that a family $\mathcal{F} \subset X$ of irreducible representations of $A = \mathcal{C}_0(X)$ is faithful if, and only if, \mathcal{F} is *dense* in X . On the other hand, a family of irreducible representations of $A = \mathcal{C}_0(X)$ is full if, and only if, $\mathcal{F} = X$.

The relation between full and exhausting families is not so simple. We begin with the following remark on the above example.

Remark 3.5. If X is moreover *metrisable*, then every exhausting family $\mathcal{F} \subset X$ is also full, because for any $x \in X$, there exists a compactly supported, continuous function $\psi_x : X \rightarrow [0, 1]$ such that $\psi_x(x) = 1$ and $\psi_x(y) < 1$ for $y \neq x$ (we can do that by arranging that $\psi_x(y) = 1 - d(x, y)$, for $d(x, y)$ small, and use the Tietze extension theorem).

In general, however, it is not true that any exhausting family is full. Indeed, let I be an uncountable set and $X = [0, 1]^I$. Let $x \in X$ be arbitrary, then the family $\mathcal{F} := X \setminus \{x\}$ is exhausting but is not full. See also Propositions ??.

We now explain how the concepts of full and exhausting sets of representations are useful for invertibility questions. Let us first notice that if an element $a \in A$ is not invertible, then either a^*a or aa^* are not invertible. This is seen for example by assuming that $A \subset \mathcal{L}(\mathcal{H})$. Assume the contrary. Then there exist y, z such that $ya^*a = 1$ and $aa^*z = 1$. So $b := ya^*a = ya^*aa^*z = a^*z$ satisfies $ab = ba = 1$, and hence a is invertible.

Theorem 3.6. *Let \mathcal{F} be a set of representations of a unital C^* -algebra A . The following are equivalent:*

- (i) \mathcal{F} is exhausting.
- (ii) An element $a \in A$ is invertible if, and only if, $\phi(a)$ is invertible in $\mathcal{L}(\mathcal{H}_\phi)$ for all $\phi \in \mathcal{F}$.

Proof. Let us assume (i) and let $a \in A$ be such that $\phi(a)$ is invertible for all $\phi \in \mathcal{F}$. We want to show that a is invertible as well. Let us assume, by contradiction, that it is not invertible. Then either a^*a or aa^* is not invertible. By replacing a with a^* (which is also not invertible), we can assume that a^*a is not invertible. Then $0 \in \text{Spec}(a^*a)$ and hence the element $b := \|a\|^2 - a^*a$ has norm $\|b\| = \|a\|^2$. Therefore there exists $\phi \in \mathcal{F}$ such that $\|\phi(b)\| = \|b\|$, since we have assumed that \mathcal{F} is exhausting. Therefore $\|a\|^2 - \phi(a)^*\phi(a) = \phi(b)$ has norm $\|b\| = \|a\|^2$, and hence 0 is in the spectrum of $\phi(a)^*\phi(a)$, which is then not invertible. Therefore $\phi(a)$ is not invertible. We have thus obtained a contradiction.

Conversely, let us assume (ii) and let $a \in A$. We want to show that there exists $\phi \in \mathcal{F}$ such that $\|\phi(a)\| = \|a\|$. Let us consider again $b := \|a\|^2 - a^*a$, which is not invertible in A , by the properties of functional calculus. Therefore, there exists $\phi \in \mathcal{F}$ such that $\phi(b) = \|a\|^2 - \phi(a)^*\phi(a)$ is not invertible. Since $\|\phi(a)\| \leq \|a\|$, this is possible only if $\|\phi(a)\| = \|a\|$. \square

The following proposition is the analog of Proposition ?? in the framework of exhausting families.

Theorem 3.7. *Let \mathcal{F} be a family of representations of a unital C^* -algebra A . Then \mathcal{F} is exhausting if, and only if,*

$$(10) \quad \text{Spec}(a) = \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a)).$$

for any $a \in A$.

Proof. Let us assume first that the family \mathcal{F} is exhausting. We proceed in analogy with the proof of Proposition ???. Since we have that $\text{Spec}(\phi_0(a)) \subset \text{Spec}(a)$ for any representation ϕ_0 of A , it is enough to show that $\text{Spec}(a) \subset \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))$. Let us assume the contrary and let $\lambda \in \text{Spec}(a) \setminus \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))$. By replacing a with $a - \lambda$, we can assume that $\lambda = 0$. We thus have that $\phi(a)$ is invertible for all $\phi \in \mathcal{F}$, but a is not invertible (in A). This is however a contradiction by Theorem ???, which implies that a must be invertible in A as well.

To prove the converse, let us assume that $\text{Spec}(a) \subset \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))$ for all $a \in A$. We shall use Theorem ???. Let us assume that $a \in A$ and that $\phi(a)$ is invertible for all $\phi \in \mathcal{F}$. Then $0 \notin \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))$. Since $\text{Spec}(a) \subset \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))$, we have that $0 \notin \text{Spec}(a)$, and hence a is invertible. Theorem ??? then shows that the family \mathcal{F} is exhausting. \square

4. TOPOLOGY ON THE SPECTRUM AND EXHAUSTING FAMILIES

Let us discuss now in more detail the relation between the concept of exhausting family and the simpler (to check) concept of a full family. The following theorem describes the class of C^* -algebras for which every exhausting family is also full. It explains Example ??? and Remark ???.

Lemma 4.1. *Let A be a C^* -algebra, J a two-sided ideal, and π a representation of A such that $\pi|_J$ is non-degenerate. Also let $a \in A$, $0 \leq a \leq 1$ such that $\|\pi(a)\| = 1$ and $\eta > 0$. Then there exists $c \in J$, $c \geq 0$, $\|c\| \leq \eta$ such that $\|\pi(a + c)\| \geq 1 + \eta/2$.*

Proof. For any fixed $\varepsilon > 0$ there exists a unit vector ξ such that $\langle \pi(a)\xi, \xi \rangle \geq 1 - \varepsilon$. Then the positive form $\varphi(b) := \langle \pi(b)\xi, \xi \rangle$ has norm $\|\varphi\| \leq \|\xi\|^2 = 1$. But if (u_λ) is an approximate unit in J , then

$$\|\varphi\| \geq \|\varphi|_J\| = \lim \varphi(u_i) = \|\xi\| = 1.$$

So $\|\varphi|_J\| = 1$ (and $\|\varphi\| = 1$ also). Hence there exists $c_0 \in J$, $c_0 \geq 0$, $\|c_0\| = 1$, such that $\varphi(c_0) \geq 1 - \varepsilon$. We then set $c = \eta c_0$ and indeed, for ε small enough

$$\|a + c\| \geq \varphi(a + c) \geq 1 - \varepsilon + \eta(1 - \varepsilon) \geq 1 + \eta/2.$$

This completes the proof. \square

Theorem 4.2. *Let A be a C^* -algebra. Let us assume that every $\pi \in \hat{A}$ has a countable base for its system of neighborhoods. Then every exhausting family \mathcal{F} of representations of A is also full.*

Conversely, if every exhausting family \mathcal{F} of representations of A is also full, then every $\pi \in \hat{A}$ has a countable base for its system of neighborhoods.

Proof. Let us assume that every $\pi \in \hat{A}$ has a countable base for its system of neighborhoods and let \mathcal{F} be an exhausting family of representations of A . Let us assume that \mathcal{F} is not full. Then there exists $\pi_0 \in \hat{A} \setminus \cup_{\phi \in \mathcal{F}} \text{supp}(\phi)$. Let

$$V_0 \supset \dots \supset V_n \supset V_{n+1} \dots \supset \{\pi_0\} = \cap_k V_k$$

be a basis for the system of neighborhoods of π in \hat{A} . We may assume that the V_n consist of open sets, $V_n = \hat{J}_n$. We shall construct $a \in A$ such that $\|a\| = \|\pi_0(a)\| = 1$, but $\|\pi(a)\| \leq 1 - 2^k$ for any $\pi \in \hat{A} \setminus V_k$. Assuming that we have found $a \in A$ with this property, then for every $\phi \in \mathcal{F}$, we have that $\hat{A} \setminus \text{supp}(\phi)$ is a neighborhood of π_0 in \hat{A} . Therefore there exists n such that $V_n \subset \hat{A} \setminus \text{supp}(\phi)$ and hence $\|\pi(a)\| \leq 1 - 2^{-n}$ for all $\pi \in \text{supp}(\phi)$. This gives $\|\phi(a)\| \leq 1 - 2^{-n} < 1$, thus contradicting the fact that \mathcal{F} is exhausting.

To construct $a \in A$ with the desired properties, let us consider the ideals J_n defining the sets V_n : $V_n = \hat{J}_n$. We shall define by induction $a_n \in A$ with the following properties:

- (i) $0 \leq a_n \leq 1$;
- (ii) $\|\pi_0(a_n)\| = 1$;
- (iii) $\|\pi_0(a_n)\| \leq 1 - 2^{-k}$ on $\hat{A} \setminus \hat{J}_k$ for $k = 0, 1, \dots, n$;
- (iv) $\|a_n - a_{n-1}\| \leq 2^{2-n}$ for $n \geq 1$.

We define a_0 as follows. We first choose $b_0 \in J_0$ such that $0 \leq b_0$, and $\pi_0(b_0) \neq 0$. By rescaling b_0 with a positive factor, we can assume that $\|\pi_0(b_0)\| = 1$. Let then $\chi_0 : [0, \infty) \rightarrow [0, 1]$ be the continuous function defined by $\chi_0(t) = t$ for $t \leq 1$ and $\chi_0(t) = 1$ for $t \geq 1$. Then we define $a_0 = \chi_0(b_0)$. Conditions (i–iii) are then satisfied.

To define a_n in terms a_{n-1} , we first define c_n and $b_n = a_{n-1} + c_n$ as follows. By lemma ?? there exists $c_n \in J_n$, $c_n \geq 0$, $\|c_n\| \leq 2^{1-n}$, such that $\|\pi_0(b_n)\| \geq 1 + 2^{-n}$. Let then $\chi_n : [0, \infty) \rightarrow [0, 1]$ be the continuous function defined by $\chi_n(t) = t$ for $t \leq 1 - 2^{1-n}$, χ_n linear on $[1 - 2^{1-n}, 1]$ and on $[1, 1 + 2^{-n}]$, $\chi_n(1) = 1 - 2^{-n}$, and $\chi_n(t) = 1$ for $t \geq 1 + 2^{-n}$. Then we define $a_n = \chi_n(b_n)$.

Let us check that conditions (i–iv) are satisfied by a_n :

- (i) We have that $a_{n-1}, c_n \geq 0$, hence $b_n := a_{n-1} + c_n \geq 0$. Since $0 \leq \chi_n \leq 1$, we obtain that $0 \leq a_n := \chi_n(b_n) \leq 1$.
- (ii) Since $\|\pi_0(b_n)\| \geq 1 + 2^{-n}$ and $\chi_n(t) = 1$ for $t \geq 1 + 2^{-n}$, we have that $\|\chi(\pi_0(b_n))\| = \|\pi_0(\chi(b_n))\| = \|\pi_0(a_n)\| = 1$.
- (iii) Let $\pi \in \hat{J}_k^c := \hat{A} \setminus \hat{J}_k$, for some $k \leq n$. Then $\pi(c_n) = 0$. Assume $k < n$. Then $\|\pi(a_{n-1})\| \leq 1 - 2^{-k} \leq 1 - 2^{1-n}$, by the induction hypothesis and hence

$$\pi(a_n) = \pi(\chi_n(b_n)) = \chi_n(\pi(b_n)) = \chi_n(\pi(a_{n-1})) = \pi(a_{n-1}),$$

since $\chi(t) = t$ for $t \leq 1 - 2^{1-n}$. Consequently, $\|\pi(a_n)\| = \|\pi(a_{n-1})\| \leq 1 - 2^{-k}$ (for $k < n$). If $k = n$, then, similarly, $\pi(a_n) = \chi_n(\pi(a_{n-1}))$. Since $\chi_n(t) \leq 1 - 2^{-n}$ for $t \leq 1$ and $0 \leq a_{n-1} \leq 1$, we have that $\|\pi(a_n)\| = \|\chi_n(\pi(a_{n-1}))\| \leq 1 - 2^{-n}$.

(iv) We have $\|b_n\| \leq \|a_{n-1}\| + \|c_n\| \leq 1 + 2^{1-n}$. Since $|\chi(t) - t| \leq 2^{1-n}$ for all t , we have $\|a_n - b_n\| \leq 2^{1-n}$. Hence $\|a_n - a_{n-1}\| \leq \|a_n - b_n\| + \|c_n\| \leq 2^{2-n}$.

We are ready now to construct our element $a \in A$. Indeed, condition (iv) satisfied by the sequence a_n shows that it is convergent. Let $a = \lim a_n$. Conditions (i–iii) are compatible with limits, hence

- (i) $0 \leq a \leq 1$;
- (ii) $\|\pi_0(a)\| = 1$;
- (iii) $\|\pi_0(a)\| \leq 1 - 2^{-k}$ on $\hat{A} \setminus \hat{J}_k$ for $k = 0, 1, \dots$

Thus a has the desired properties, and hence we obtain the proof of the first part of the proposition.

The converse is easier. Let $\pi_0 \in \hat{A}$. Then $\mathcal{F} := \hat{A} \setminus \{\pi_0\}$ is not full. By our assumption, it is also not exhausting. Hence there exists $a \in A$, such that

$\|\pi(a)\| < \|a\|$ for all $\pi \in \hat{A}$, $\pi \neq \pi_0$. By rescaling, we can assume $\|a\| = \|\pi_0(a)\| = 1$. Then the sets

$$V_n := \{ \pi \in \hat{A}, \|\pi(a)\| > 1 - 2^{-n} \}$$

are open. Let us show that they form a basis for the system of neighborhoods of π_0 . Indeed, let G be an arbitrary open subset of \hat{A} containing π_0 . Then there exists a two-sided ideal $J \subset A$ such that $G = \hat{J}$. The set of irreducible representations of A/J identifies with $\hat{J}^c := \hat{A} \setminus \hat{J}$. Hence $\|\pi(a)\| < 1$ for all $\pi \in \widehat{A/J} := \hat{J}^c$, and consequently $\|a+J\| < 1$ (the norm is in A/J). Let n be such that $\|a+J\| \leq 1 - 2^{-n}$. Then $V_n \subset \hat{J}$. The proof is now complete. \square

It is easy to show that separable C^* -algebras satisfy the assumptions of the previous proposition.

Proposition 4.3. *Let A be a separable C^* -algebra. Then every irreducible representation $\pi \in \hat{A}$ has a countable base for its system of neighborhoods. Consequently, if \mathcal{F} is an exhausting set of representations of A , then \mathcal{F} is full.*

Proof. Let $\{a_n\}$ be a dense subset of A and fix $\pi_0 \in \hat{A}$. Define

$$V_n := \{ \pi \in \hat{A}, \|\pi(a_n)\| > \|\pi_0(a_n)\|/2 \}.$$

Then V_n is open by Lemma ???. We claim that V_n is a basis of the system of neighborhoods of π_0 in \hat{A} . Indeed, let $G \subset \hat{A}$ be an open set containing π_0 . Then $G = \hat{J}$ for some two-sided ideal of A and $\pi_0 \neq 0$ on J . Let $a \in J$ such that $\pi_0(a) \neq 0$. We can find a_n such that $\|a - a_n\| < \|\pi_0(a)\|/4$. Then $\|\pi'(a) - \pi'(a_n)\| < \|\pi_0(a)\|/4$ for any $\pi' \in \hat{A}$, and hence

$$(11) \quad \|\pi'(a)\| - \|\pi_0(a)\|/4 < \|\pi'(a_n)\| < \|\pi'(a)\| + \|\pi_0(a)\|/4, \quad \forall \pi' \in \hat{A}.$$

To show that $V_n \subset G$, it is enough to show that $V_n \cap G^c = V_n \cap \hat{J}^c = \emptyset$. Suppose the contrary and let $\pi \in V_n \cap \hat{J}^c$. Then $\|\pi(a_n)\| > \|\pi_0(a_n)\|/2$ and $\pi(a) = 0$. Let us show that this is not possible. Indeed, using also Equation (??) for $\pi' = \pi_0$ and $\pi' = \pi$, we obtain

$$\frac{3}{8}\|\pi_0(a)\| < \frac{1}{2}\|\pi_0(a_n)\| < \|\pi(a_n)\| < \frac{1}{4}\|\pi_0(a)\|,$$

which is contradiction. Consequently $V_n \subset G$ and $\{V_n\}$ is a basis for the system of neighborhoods of π_0 in \hat{A} , as claimed. \square

The next two basic examples illustrate the differences between the notions of faithful and exhausting families.

Example 4.4. Let A be the C^* -algebra of continuous functions f on $[0, 1]$ with values in $M_2(\mathbb{C})$ such that $f(1)$ is diagonal. Then the maps $ev_t: f \mapsto f(t) \in M_2(\mathbb{C})$, for $t < 1$, together with the maps $ev_1^i: f \mapsto f(1)_{ii}$ ($i = 0, 1$) provide all the irreducible representations of A (up to equivalence). The family

$$\mathcal{F} = \{ ev_t, t < 1 \} \cup \{ ev_1^1 \}$$

is a faithful but not full family. In fact the function $t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1-t \end{pmatrix}$ is not invertible in A but $\pi(f)$ is invertible for all $\pi \in \mathcal{F}$. Of course, in this example, every $\pi \in \hat{A}$ has a countable base for its system of neighborhoods, so every exhausting family of representations \mathcal{F} of A is also full.

Example 4.5. The next example is closely related to the examples we will be dealing with below. Let \mathcal{T} be the Toeplitz algebra. It is the C^* -algebra generated by the operator defined by the unilateral shift S . (Recall that S acts on the Hilbert space $L^2(\mathbb{N})$ by $S: \epsilon_k \mapsto \epsilon_{k+1}$. As $S^*S = 1$ and $SS^* - 1$ is a rank 1 operator, one gets the exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{T} \rightarrow C(S^1) \rightarrow 0$$

where \mathcal{K} is the algebra of compact operator. Extend the unique irreducible representation π of \mathcal{K} to \mathcal{T} as in [?]. Also irreducible characters χ_θ of S^1 pull-back to irreducible characters of \mathcal{T} vanishing on \mathcal{K} . Then the spectrum of \mathcal{T} is

$$\widehat{\mathcal{T}} = \{\pi\} \cup \{\chi_\theta, \theta \in S^1\},$$

with S^1 embedded as a closed subset. A subset $V \subset \widehat{\mathcal{T}}$ will be open if, and only if, it contains π and its intersection with S^1 is open. We thus see that the single element set $\{\pi\}$ defines a full family. In other words $\widehat{\mathcal{T}} = \overline{\{\pi\}} = \text{supp}(\pi)$. In this example again every $\pi \in \widehat{A}$ has a countable base for its system of neighborhoods, so every exhausting family of representations \mathcal{F} of A is also full. We can see directly that the family $\mathcal{F} = \{\pi\}$ (consisting of π alone) is exhausting. Indeed, it suffices to notice that $\|x\| = \|\pi(x)\|$ for all x since π is injective.

Here are two more examples that show that the condition that A be separable is not necessary for the classes of full families of representations and exhausting families of representations to coincide.

Example 4.6. Let I be an infinite *uncountable* set. We endow it with the discrete topology. Then $A_0 := C_0(I)$ and $A_1 := \mathcal{K}(\ell^2(I))$ (the algebra of compact operators on $\ell^2(I)$) are not separable, however, if \mathcal{F} is an exhausting family of representations of A_i , $i = 0, 1$, then \mathcal{F} is also a full family of representations of A_i .

5. UNBOUNDED OPERATORS

The results of the previous sections are of interest mainly in applications to unbounded operators, so we now extend Theorem ?? to (possibly) unbounded operators affiliated to C^* -algebras. We begin with an abstract setting.

5.1. Abstract affiliated operators. The notion of affiliated self-adjoint operator has been extensively and successfully studied, see [?, ?, ?, ?, ?] for example. In the sequel we will closely follow the definitions in [?], beginning with an abstract version of this notion. See [?, ?] for results on unbounded operators on Hilbert modules [?, ?, ?].

Definition 5.1. Let A be a C^* -algebra. An *observable* T *affiliated to* A is a morphism $\theta_T : C_0(\mathbb{R}) \rightarrow A$ of C^* -algebras. The observable T is said to be *strictly affiliated to* A if the space generated by elements of the form $\theta_T(h)a$ ($a \in A$, $h \in C_0(\mathbb{R})$), is dense in A .

As in the classical case, we now introduce the Cayley transform. To this end, let us notice that an observable affiliated to A extends to a morphism $\theta_T : C_0(\mathbb{R})^+ \rightarrow A^+$ (the algebra obtained from A by adjunction of a unit). If moreover T is strictly affiliated to A , then θ_T extends to a morphism from $C_b(\mathbb{R})$ to the multiplier algebra of A , but we shall not need this fact.

Definition 5.2. Let T be an observable affiliated to A . The *Cayley transform* $u_T \in A^+$ of T is

$$(12) \quad u_T := \theta_T(h_0), \quad h_0(z) := (z + \iota)(z - \iota)^{-1}.$$

The Cayley transform allows us to reduce questions about the spectrum of an observable to questions about the spectrum of its Cayley transform. Let us first introduce, however, the *spectrum of an affiliated observable*. Let thus $\theta_T : \mathcal{C}_0(\mathbb{R}) \rightarrow A$ be a self-adjoint operator affiliated to a C^* -algebra A . The kernel of θ_T is then of the form $\mathcal{C}_0(U)$, for some open subset of \mathbb{R} . We define the spectrum $\text{Spec}_A(T)$ as the complement of U in \mathbb{R} . Explicitly,

$$(13) \quad \text{Spec}_A(T) = \{\lambda \in \mathbb{R}, h(\lambda) = 0, \forall h \in \mathcal{C}_0(\mathbb{R}) \text{ such that } \theta_T(h) = 0\}.$$

If $\sigma : A \rightarrow B$ is a morphism of C^* -algebras, then $\sigma \circ \theta_T : \mathcal{C}_0(\mathbb{R}) \rightarrow B$ is an observable $\sigma(T)$ affiliated to the C^* -algebra B and

$$\text{Spec}_B(\sigma(T)) \subset \text{Spec}_A(T).$$

If σ is injective, then $\text{Spec}_B(\sigma(T)) = \text{Spec}_A(T)$, which shows that the spectrum is preserved by increasing the C^* -algebra A . Note that

$$(14) \quad \sigma(u_T) = u_{\sigma(T)}.$$

If $\text{Spec}(T)$ is a bounded subset of \mathbb{R} , then we shall say that T is *bounded*, in which case, we can define a true operator $T := \iota(u_T + 1)(u_T - 1)^{-1} \in A$ such that $\theta_T(h) = h(T)$, $h \in \mathcal{C}_0(\mathbb{R})$. Otherwise, we shall say that T is *unbounded*. Let $h_0(z) := (z + \iota)(z - \iota)^{-1}$, as before.

Lemma 5.3. *The spectrum $\text{Spec}(T)$ of the an observable $\theta_T : \mathcal{C}_0(\mathbb{R}) \rightarrow A$ affiliated to the C^* -algebra A and the spectrum $\text{Spec}(u_T)$ of its Cayley transform are related by*

$$\text{Spec}(T) = h_0^{-1}(\text{Spec}(u_T)).$$

Proof. This follows from the fact that h_0 is a homeomorphism of \mathbb{R} onto its image in $S^1 := \{|z| = 1\}$ and from the properties of the functional calculus. \square

One can make the relation in the above lemma more precise by saying that for bounded T we have $h_0(\text{Spec}(T)) = \text{Spec}(u_T)$ whereas for unbounded T we have

$$(15) \quad \overline{h_0(\text{Spec}(T))} = h_0(\text{Spec}(T)) \cup \{1\} = \text{Spec}(u_T),$$

where $h_0(z) := (z + \iota)(z - \iota)^{-1}$, as before.

Here is our main result on (possibly unbounded) self-adjoint operators affiliated to C^* -algebras.

Theorem 5.4. *Let A be a unital C^* -algebra and T an observable affiliated to A . Let \mathcal{F} be a set of representations of A .*

(1) *If \mathcal{F} is exhausting, then*

$$\text{Spec}(T) = \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(T)).$$

(2) *If \mathcal{F} is faithful, then*

$$\text{Spec}(T) = \overline{\cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(T))}.$$

Proof. The proofs of (i) and (ii) are similar, starting with the relation $\text{Spec}(T) = h_0^{-1}(\text{Spec}(u_T))$ of Lemma ???. We begin with (i), which is slightly easier. Since \mathcal{F} is exhausting, we can then apply theorem ?? to $u_T \in A^+$ and the family $\sigma \in \mathcal{F}$. We obtain

$$\begin{aligned} \text{Spec}(T) &= h_0^{-1}[\text{Spec}(u_T)] = h_0^{-1}[\cup_{\sigma \in \mathcal{F}} \text{Spec}(\sigma(u_T))] \\ &= h_0^{-1}[\cup_{\sigma \in \mathcal{F}} \text{Spec}(u_{\sigma(T)})] = \cup_{\sigma \in \mathcal{F}} h_0^{-1}[\text{Spec}(u_{\sigma(T)})] = \cup_{\sigma \in \mathcal{F}} \text{Spec}(\sigma(T)). \end{aligned}$$

If, on the other hand, \mathcal{F} is faithful, we apply proposition ?? after noting that h_0 is a homeomorphism of \mathbb{R} onto its image in $S^1 := \{|z| = 1\}$ and hence $h_0^{-1}(\overline{S}) = \overline{h_0^{-1}(S)}$ for any $S \subset S^1$. The same argument then gives

$$\begin{aligned} \text{Spec}(T) &= h_0^{-1}[\text{Spec}(u_T)] = h_0^{-1}[\overline{\cup_{\sigma \in \mathcal{F}} \text{Spec}(\sigma(u_T))}] \\ &= h_0^{-1}[\overline{\cup_{\sigma \in \mathcal{F}} \text{Spec}(u_{\sigma(T)})}] = \overline{\cup_{\sigma \in \mathcal{F}} h_0^{-1}[\text{Spec}(u_{\sigma(T)})]} = \overline{\cup_{\sigma \in \mathcal{F}} \text{Spec}(\sigma(T))}. \end{aligned}$$

The proof is now complete. \square

5.2. The case of true operators. We now look at concrete (true) operators.

Definition 5.5. Let $A \subset \mathcal{L}(\mathcal{H})$ be a sub- C^* -algebra of $\mathcal{L}(\mathcal{H})$. A (possibly unbounded) self-adjoint operator $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}$ is said to be *affiliated* to A if, for every continuous functions h on the spectrum of T vanishing at infinity, we have $h(T) \in A$.

Remark 5.6. We have that T is affiliated to A if, and only if, $(T - \lambda)^{-1} \in A$ for one $\lambda \notin \text{Spec}(T)$ (equivalently for all such λ) [?]. We thus see that a self-adjoint operator T affiliated to A defines a morphism $\theta_T : \mathcal{C}_0(\mathbb{R}) \rightarrow A$, $\theta_T(h) := h(T)$ such that $\text{Spec}(T) = \text{Spec}(\theta_T)$. Thus T defines an observable affiliated to A .

When $A \subset \mathcal{L}(\mathcal{H})$ is non degenerate, the correspondence between self-adjoint operators affiliated to A and observables affiliated to A given by $T \mapsto \theta_T$ is actually bijective. This can be seen by using the unbounded functional calculus of normal operators.

Since in our paper we shall consider only the case when $A \subset \mathcal{L}(\mathcal{H})$ is non degenerate, we shall not make a difference between operators and observables affiliated to A .

Recall that an unbounded operator T is invertible if, and only if, it is bijective and T^{-1} is bounded. This is also equivalent to $0 \notin \text{Spec}(\theta_T)$. We have the following analog of Proposition ?? and Theorem ??

Theorem 5.7. *Let $A \subset \mathcal{L}(\mathcal{H})$ be a unital C^* -algebra and T a self-adjoint operator affiliated to A . Let \mathcal{F} be a set of representations of A .*

- (1) *Let \mathcal{F} be exhausting. Then T is invertible if, and only if $\phi(T)$ is invertible for all $\phi \in \mathcal{F}$.*
- (2) *Let \mathcal{F} be faithful. Then T is invertible if, and only if $\phi(T)$ is invertible for all $\phi \in \mathcal{F}$ and the set $\{\|\phi(T)^{-1}\|, \phi \in \mathcal{F}\}$ is bounded.*

Proof. This follows from Theorem ?? as follows. First of all, we have that T is invertible if, and only if, $0 \notin \text{Spec}(T)$. Now, if \mathcal{F} is exhausting, we have

$$0 \notin \text{Spec}(T) \Leftrightarrow 0 \notin \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(T)) \Leftrightarrow 0 \notin \text{Spec}(\phi(T)) \text{ for all } \phi \in \mathcal{F}.$$

This proves (i).

To prove (ii), we proceed similarly, noticing also that the distance from 0 to $\text{Spec}(T)$ is exactly $\|T^{-1}\|$. \square

6. PARAMETRIC PSEUDODIFFERENTIAL OPERATORS

Let M be a compact smooth Riemannian manifold and G be a Lie group (finite dimensional). Let $\Psi^0(M \times G)^G$ be the C^* -algebra of order 0, G -invariant pseudodifferential operators on $M \times G$ and $A := \overline{\Psi^0(M \times G)^G}$ be its norm closure acting on $L^2(M \times G)$. We get the following exact sequence [?, ?, ?]

$$(16) \quad 0 \rightarrow C_r^*(G) \otimes \mathcal{K} \rightarrow \overline{\Psi^0(M \times G)^G} \rightarrow \mathcal{C}_0(S^*(M \times \text{Lie}(G))) \rightarrow 0.$$

Note that the kernel of the symbol map will now have irreducible representations parametrised by the unitary irreducible representations of G . Let $T \in \Psi^m(M \times G)^G$. Then $T^* \in \Psi^m(M \times G)^G$ is defined using the properties of pseudodifferential operators. All operators considered below are closed with minimal domain.

Lemma 6.1. *If $T = T^*$ and T is elliptic, then T^* is actually the Hilbert space adjoint of T , so T is self-adjoint. Moreover, T satisfies $(T + i)^{-1} \in C_r^*(G)$, and hence it is affiliated to $C_r^*(G)$.*

Proof. This was proved in the papers [?, ?, ?]. \square

In other words, any elliptic, formally self-adjoint $T \in \Psi^m(M \times G)^G$, $m > 0$, is actually self-adjoint.

Let us assume $G = \mathbb{R}^n$, regarded as an abelian Lie group. Then our exact sequence (??) becomes

$$(17) \quad 0 \rightarrow \mathcal{C}_0(\mathbb{R}^n) \otimes \mathcal{K} \rightarrow \overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}} \rightarrow \mathcal{C}_0(S^*(M \times \mathbb{R}^n)) \rightarrow 0.$$

Then we use that for each $\lambda \in \mathbb{R}^n$, there corresponds an irreducible representation ϕ_λ of $\mathcal{C}_0(\mathbb{R}^n) \otimes \mathcal{K}$ that extends uniquely to an irreducible representation of $\overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}}$ denoted by the same letter. (We have used here the fact that an irreducible representation ϕ_0 of an ideal I of a C^* -algebra A extends uniquely to an irreducible representation of A .) It is customary to denote by $\hat{T}(\lambda) := \phi_\lambda(T)$ for T a pseudodifferential operator in $\Psi^m(M \times \mathbb{R}^n)^{\mathbb{R}^n}$, $m \geq 0$. To define $\hat{T}(\lambda)$ for $m > 0$, we can either use the Fourier transform or, notice that Δ is affiliated to the closure of $\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}$. This allows us to define $\hat{\Delta}(\lambda)$. In general, we write $T = (1 - \Delta)^k S$, with $S \in \Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}$ and define $\hat{T} = \widehat{(1 - \Delta)}(\lambda)^k \hat{S}(\lambda)$. (We consider the ‘‘analyst’s’’ Laplacian, so $\Delta \leq 0$.)

Proposition 6.2. *Let us denote $\mathcal{F} = \{\phi_\lambda, \lambda \in \mathbb{R}^n\}$.*

- (i) *The family \mathcal{F} is a full family of representations of $\overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}}$.*
- (ii) *Let $P \in \Psi^m(M \times \mathbb{R}^n)^{\mathbb{R}^n}$, then $P : H^s(M \times \mathbb{R}^n) \rightarrow H^{s-m}(M \times \mathbb{R}^n)$ is invertible if, and only if $\hat{P}(\lambda) : H^s(M) \rightarrow H^{s-m}(M)$ is invertible for all $\lambda \in \mathbb{R}^n$*
- (iii) *If $T \in \Psi^m(M \times \mathbb{R}^n)^{\mathbb{R}^n}$ is formally self-adjoint, then*

$$\text{Spec}(T) = \cup_{\lambda \in \mathbb{R}^n} \text{Spec}(\hat{T}(\lambda)).$$

Proof. Let us prove first (i). Let us denote $A := \overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}}$, for simplicity. The primitive ideal spectrum \hat{A} of the C^* -algebra A is given by $\mathbb{R}^n \cup S^*(TM \times \mathbb{R}^n)$, with the first copy of \mathbb{R}^n corresponding to the representations ϕ_λ of A that do not vanish on $\mathcal{C}_0(\mathbb{R}^n) \otimes \mathcal{K}$. The closure of ϕ_λ consists of the union of $\{\lambda\}$ with the set $S^*(TM \times \mathbb{R}^n) \cap \{(\xi, \lambda) \in T^*M \times \mathbb{R}^n\}$. The family $\{\phi_\lambda\}$ is therefore full.

To prove (ii), let us denote by $\Delta_M \leq 0$ the (non-positive) Laplace operator on M . Then the Laplace operator Δ on $M \times \mathbb{R}^n$ is $\Delta = \Delta_{\mathbb{R}^n} + \Delta_M$. Note that $(1 - \Delta)^{m/2} : H^s(M \times \mathbb{R}^n) \rightarrow H^{s-m}(M \times \mathbb{R}^n)$ and $(c - \Delta_M)^{m/2} : H^s(M) \rightarrow H^{s-m}(M)$, $c > 0$, are isomorphisms. By [?], we have that

$$P_1 := (1 - \Delta)^{(s-m)/2} P (1 - \Delta)^{-s/2} \in A := \overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}}.$$

It is then enough to prove that P_1 is invertible on $L^2(M \times \mathbb{R}^n)$. Moreover from part (i) we have just proved and Theorem ?? we know that P_1 is invertible on $L^2(M \times \mathbb{R}^n)$ if, and only if, $\hat{P}_1(\lambda) := \phi_\lambda(P_1)$ is invertible on $L^2(M)$ for all $\lambda \in \mathbb{R}^n$. But, using also $\widehat{1 - \Delta}(\lambda) = (1 + |\lambda|^2 - \Delta_M)$, we have

$$\hat{P}_1(\lambda) = (1 + |\lambda|^2 - \Delta_M)^{(s-m)/2} \hat{P}(\lambda) (1 + |\lambda|^2 - \Delta_M)^{-s/2},$$

which is invertible by assumption

To prove (iii), we recall that T is affiliated to A , by Lemma ?. The result then follows from Theorem ??(ii). \square

Operators of the kind considered in this subsection were used also in [?, ?, ?, ?, ?, ?, ?]. They turn out to be useful also for general topological index theorems [?, ?]. A more class of operators than the ones considered in this subsection were introduced in [?, ?]. The above result has turned out to be useful for the study of layer potentials [?].

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