A REFINED HVZ-THEOREM FOR ASYMPTOTICALLY HOMOGENEOUS INTERACTIONS AND FINITELY MANY COLLISION PLANES

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ABSTRACT. We study algebras associated to N-body type Hamiltonians with interactions that are asymptotically homogeneous at infinity on a finite dimensional, vector real space X. More precisely, let $Y \subset X$ be a linear subspace and v_Y be a continuous function on X/Y that has uniform homogeneous radial limits at infinity. We consider in this paper Hamiltonians of the form $H = -\Delta + \sum_{Y \in \mathcal{S}} v_Y$, where the subspaces $Y \subset X$ belong to some given, semi-lattice $\mathcal{S}$ of subspaces of X. Georgescu and Nistor have considered the case when $\mathcal{S}$ consists of *all* subspaces $Y \subset X$ (in a paper to appear in Journal of Operator Theory). As in that paper, we also consider more general Hamiltonians affiliated to a suitable cross-product algebra $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$. A first goal of this note is to see which results of that paper carry through to the case when $\mathcal{S}$ (the set of "collision planes") is finite and, for the ones that do not, what is their suitable modification. While the results on the essential spectra of the resulting Hamiltonians and the affiliation criteria carry through, the spectra of the corresponding algebras are quite different. Identifying these spectra may have implications for regularity of eigenvalues and numerical methods. Our results also shed some new light on the results of Georgescu and Nistor in the aforementioned paper and, in general, on the theory developed by Georgescu and his collaborators. For instance, we show that, in our case, the closure is not needed in the union of the spectra of the limit operators. We also give a quotient topology description of the topology on the spectrum of the *graded* N-body C^*-algebras introduced by Georgescu.

CONTENTS

1. INTRODUCTION

We continue the study begun by Georgescu and Nistor [14] of Hamiltonians of N-body type with interactions that are asymptotically homogeneous at infinity on a finite dimensional Euclidean space X. The Hamiltonians considered in that paper were obtained by a procedure (described below) that was employing *all* subspaces $Y \subset X$, whereas in this paper, we only consider those subspaces Y that belong to a suitable semi-lattice $\mathcal{S}$ of

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subspaces of X satisfying $X \in \mathcal{S}$. (Thus $Z_1 \cap Z_2 \in \mathcal{S}$ if $Z_1, Z_2 \in \mathcal{S}$.) Whenever possible, we follow the broad lines of [14]. Eventually, we shall assume that $\mathcal{S}$ is finite, but we begin with the general case.

To fix ideas, let us mention right away an important example of a semi-lattice that arises in the study of quantum N-body problems. Namely, it is the semi-lattice $\mathcal{S}_N$ of subspaces of $X := \mathbb{R}^{3N}$ generated by the subspaces X and

$$\begin{aligned} \mathcal{P}_j &:= \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^{3N} \mid x_j = 0 \in \mathbb{R}^3\}\,, \quad 1 \le j \le N\,, \ \text{ and} \\ \mathcal{P}_{ij} &:= \{(x_1, x_2, \ldots, x_N) \in \mathbb{R}^{3N} \mid x_i = x_j \in \mathbb{R}^3\}\,, \quad 1 \le i < j \le N\,. \end{aligned} \tag{1}$$

Thus, in addition to the spaces X, $\mathcal{P}_j$, and $\mathcal{P}_{ij}$, the semi-lattice $\mathcal{S}_N$ (the *N-body semi-lattice*) contains also all intersections of the subspaces $\mathcal{P}_j$ and $\mathcal{P}_{ij}$.

Let us fix a semi-lattice $\mathcal{S}$ with $X \in \mathcal{S}$. It turns out that the results in [14] on essential spectra and on the affiliation of operators carry through to this arbitrary semi-lattice $\mathcal{S}$. This is easy to see and is explained in this introduction. However, some important intermediate results on the representations of the cross-product algebras $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$ that control the Fredholm property, are different in the general case. (See Equation (3) for the definition of the algebra $\mathcal{E}_{\mathcal{S}}(X)$.) A careful study of the representations of these algebras also allows us to sharpen the results on the essential spectra by removing the closure in the union of the spectra of the limit operators when $\mathcal{S}$ is finite. (See Theorem 1.1.)

Possible applications of the extensions presented in this paper are to regularity results and hence to numerical methods for the resulting Hamiltonians and the study of the fine structure of their spectrum. Some of the applications and proofs will be included in a forthcoming paper, here concentrating instead on the global picture. Nevertheless, we include the proofs of some results that we are not planing to discus anywhere else, such as the topology on the spectrum of Georgescu's graded algebras. We also include complete details of the proof that we can remove the closure in the union of the spectra in Theorem 1.1. The proof of this result may also be useful for other applications.

Let us now discuss the settings of the paper and state our first result on essential spectra, Theorem 1.1.

For any real vector space Z, we let $\overline{Z}$ denote its spherical compactification (this standard notion is discussed in great detail in [14]). A function in $\mathcal{C}(\overline{Z})$ is thus a continuous function on Z that has uniform radial limits at infinity. For any subspace $Y \subset X$, $\pi_Y : X \to X/Y$ denotes the canonical projection. Let

$$H := -\Delta + \sum_{Y \in \mathcal{S}} v_Y\,, \tag{2}$$

where $v_Y \in \mathcal{C}(\overline{X/Y})$ is regarded also as a function on X via the projection $\pi_Y : X \to X/Y$. The sum is over *all* subspaces $Y \subset X$, $Y \in \mathcal{S}$, but is assumed to be convergent. One of the main result of [14] describes, in particular, the essential spectrum of H on $L^2(X)$ when $\mathcal{S}$ consists of *all* subspaces of X as $\sigma_{\text{ess}}(H) = \overline{\cup}_{\alpha \in \mathbb{S}_X} \sigma(\tau_\alpha(H))$, with the notation being the one used in Theorem 1.1. This result directly extends the celebrated HVZ theorem [4, 29, 33]. A first goal of this paper is to explain how the results and methods of [14] are affected by assuming that $\mathcal{S}$ is finite. We include also some extensions of the results in [14].

On a technical level, we obtain smaller algebras than the ones in [14], so the results on the affiliation of operators and on their essential spectra remain valid in our case in a trivial way (this is because, if $B \subset A$ and we have results for operators affiliated to A, then these results will be valid for operators affiliated to B as well). We will thus review just a small sample of results of this kind. On the other hand, for a possible further study of Hamiltonians of the form (2), it may be useful to have an explicit description of the spectra of the intermediate algebras involved (the algebras $\mathcal{E}_{\mathcal{S}}(X)$ and $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$ introduced next). These spectra change dramatically in the case $\mathcal{S}$ finite. Concretely, let

$$\mathcal{E}_{\mathcal{S}}(X) \ := \ \langle \mathcal{C}(\overline{X/Y}) \rangle \,, \quad Y \in \mathcal{S} \,. \tag{3}$$

In other words, $\mathcal{E}_{\mathcal{S}}(X)$ is the closure in norm of the algebra of functions on X generated by all functions of the form $u \circ \pi_Y$, where $Y \in \mathcal{S}$ and $u \in \mathcal{C}(\overline{X/Y})$. Since X acts continuously by translations on $\mathcal{E}_{\mathcal{S}}(X)$, we can define the crossed product C^*-algebra $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$, which can be regarded as an algebra of operators on $L^2(X)$. It is the algebra generated by operators of multiplication by functions in $\mathcal{E}_{\mathcal{S}}(X)$ and operators of convolution, that is, by operators of the form $m_f C_\phi$, where m_f is the operator of multiplication by $f \in \mathcal{E}_{\mathcal{S}}(X)$ and $C_\phi u(x) := \int_X \phi(y) u(x-y) dy$ is the operator of convolution by $\phi \in \mathcal{C}_{\mathrm{c}}^\infty(X)$. Let $V \in \mathcal{E}_{\mathcal{S}}(X)$ (for instance, we could take $V := \sum_{Y \in \mathcal{S}} v_Y$, as in Equation (2)). Recalling that H and Δ are self-adjoint, we then obtain

$$(H + \mathrm{i})^{-1} \ = \ (-\Delta + \mathrm{i})^{-1} \big[1 + V(-\Delta + \mathrm{i})^{-1}\big]^{-1} \ \in \mathcal{E}_{\mathcal{S}}(X) \rtimes X \,. \tag{4}$$

This means that the operator H of Equation (2) is affiliated to $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$. (Let A be a C^*-algebra. Recall that a self-adjoint operator P *affiliated* to A is an operator P with the property that $(P + \mathrm{i})^{-1} \in A$ [6].) This is, in fact, one of the starting points of the theory developed by Georgescu and his collaborators [5, 6, 12, 13].

For each $x \in X$, we let $(T_x f)(y) := f(y - x)$ denote the translation on $L^2(X)$. Let $\mathbb{S}_X$ be the set of half-lines in X, that is

$$\mathbb{S}_X \ := \ \{\, \hat{a}, \ a \in X, a \neq 0 \,\} \,, \tag{5}$$

where $\hat{a} := \{ra \,|\, r > 0\}$. For any operator P on $L^2(X)$, we let

$$\tau_\alpha(P) \ := \ \operatorname*{s-lim}_{r \to +\infty} T_{ra}^* P T_{ra} \,, \quad \text{if } \ \alpha = \hat{a} \in \mathbb{S}_X \,, \tag{6}$$

whenever the *strong* limit exists. We identify $\mathbb{S}_Z = \overline{Z} \smallsetminus Z$ for any real vector space Z.

Theorem 1.1. *The operator H of Equation* (2) *is self-adjoint and affiliated to $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$. Let H be any self-adjoint operator affiliated to $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$ and $\alpha = \hat{a} \in \mathbb{S}_X$. Then the limit $\tau_\alpha(H) := \text{s-lim}_{r \to +\infty} T_{ra}^* H T_{ra}$ exists and, if $\mathcal{S}$ is finite, then*

$$\sigma_{\mathrm{ess}}(H) \ = \ \cup_{\alpha \in \mathbb{S}_X} \sigma(\tau_\alpha(H)) \,.$$

Most of this theorem is (essentially) contained in [14], however, in that paper, only the relation $\sigma_{\mathrm{ess}}(H) = \overline{\cup}_{\alpha \in \mathbb{S}_X} \sigma(\tau_\alpha(H))$ was proven, but without restrictions on $\mathcal{S}$. This amounts to the fact that the family $\{\tau_\alpha\}$ is a *faithful* family of representations of $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$. Our stronger result is obtained by showing that the family $\{\tau_\alpha\}$ is actually an *exhausting* family of representations of $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$ (Theorem 6.6). We notice that if $0 \notin \mathcal{S}$, then the part of the above theorem on the essential spectrum simply states that $\sigma_{\mathrm{ess}}(H) = \sigma(H)$, as H is be among the operators $\tau_\alpha(H)$, since H is invariant with respect to a non-trivial subspace of X. In the theorem, the limits of self-adjoint, unbounded operators are in strong resolvent sense, in the sense that $\text{s-lim}_{r \to +\infty} T_{ra}^* (H + \mathrm{i})^{-1} T_{ra}$ exists for each $\alpha = \mathbb{R}_+^* a$. If the limit is zero (which may happen if the potential V is unbounded at infinity), then we agree that $\tau_\alpha(H) = \infty$ and $\sigma(\tau_\alpha(H)) = \emptyset$, see [6, 23].

One of the main points of Theorem 1.1 is that the operators $\tau_\alpha(H)$ are generally simpler than H (if $0 \in \mathcal{S}$) and (often) easy to identify. The operators $\tau_\alpha(H)$ are sometimes called *limit operators*.

Here is a typical example. If $u : X \to \mathbb{C}$, we shall write $\text{av-lim}_\alpha u = c \in \mathbb{C}$ if $\lim_{a\to\alpha} \int_{a+\Lambda} |u(x) - c| \mathrm{d}x = 0$ for some (hence any!) bounded neighborhood of Λ of $0 \in X$. Here $a \in X \subset \overline{X} := X \cup \mathbb{S}_X$, $\alpha \in \mathbb{S}_X$, and the convergence is in the natural topology of the spherical compactification $\overline{X}$ of X. For instance, let us assume that we are given real valued functions v_Y, $Y \in \mathcal{S}$, such that $\text{av-lim}_\alpha v_Y$ exists for all $\alpha \in \mathbb{S}_{X/Y}$ and $v_Y = 0$ except for finitely many subspaces Y. Let $V := \sum_Y v_Y$. If $\alpha \not\subset Y$ then $\pi_Y(\alpha) \in \mathbb{S}_{X/Y}$ is a well defined half-line in the quotient and we may define $v_Y(\alpha) := \text{av-lim}_{\pi_Y(\alpha)} v_Y$. Then Proposition 1.3 of [14] gives that

$$\tau_\alpha(H) \ = \ -\Delta + \sum_{Y\supset\alpha} v_Y + \sum_{Y\not\supset\alpha} v_Y(\alpha)\,. \tag{7}$$

For the usual N-body type Hamiltonians, we have that $v_Y : X/Y \to \mathbb{R}$ vanish at infinity. In that case $\tau_\alpha(H) = -\Delta + \sum_{Y\supset\alpha} v_Y$, which is the usual version of the HVZ theorem. This calculation remains valid for operators of the form (8).

For the result of Theorem 1.1 to be effective, we need some concrete examples of self-adjoint operators on $L^2(X)$ affiliated to $\mathcal{E}_\mathcal{S}(X) \rtimes X$. Let us briefly recall the affiliation criteria of [14] and see that they work in our setting as well.

Let $\mathcal{B}(\overline{X})$ be the set of functions $u \in L^\infty(X)$ such that the "averaged limits" $\text{av-lim}_\alpha u$ (defined earlier) exist for any $\alpha \in \mathbb{S}_X$ and let $\mathcal{E}^\sharp_\mathcal{S}(X) \subset L^\infty(X)$ be the norm closed subalgebra of $L^\infty(X)$ generated by the algebras $\mathcal{B}(\overline{X/Y})$, when $Y \in \mathcal{S}$. Let h be a proper real function $h : X^* \to [0,\infty)$ (i.e. $|h(k)| \to \infty$ for $k \to \infty$). Also, let $\mathcal{F} : L^2(X) \to L^2(X^*)$ be the Fourier transform and $h(p) := \mathcal{F}^{-1} m_h \mathcal{F}$ be the associated convolution operator. We consider then $v \in L^1_{loc}(X)$ a real valued function such that there exists a sequence $v_n \in \mathcal{E}^\sharp_\mathcal{S}(X)$ of real valued functions with the property that $(1 + h(p))^{-1} v_n$ is convergent in norm to $(1 + h(p))^{-1} v$. Then

$$H \ := \ h(p) + v \tag{8}$$

is affiliated to $\mathcal{E}_\mathcal{S}(X) \rtimes X$. This allows us to consider potentials v with Coulomb type singularities (in particular, unbounded).

A second example of affiliated operators is obtained by considering symmetric, uniformly strongly elliptic operators $\sum_{|\alpha|+|\beta|\le m} \partial^\alpha g_{\alpha\beta} \partial^\beta$ with coefficients $g_{\alpha\beta} \in \mathcal{E}^\sharp_\mathcal{S}(X)$, as in [14].

See [1, 4, 8, 29, 32] for a general introduction to the basics of the problems studied here and [14] for some more specific references. In addition to the works of Georgescu and his collaborators mentioned above, essential spectra have been studied using algebraic methods by many people, including [7, 3, 16, 15, 19, 28, 26, 27, 30, 31]. We also note the similar approach to magnetic Schrödinger operators [21, 22].

We now briefly describe the contents of the paper. In Section 2, we recall the theory developed by Georgescu and his collaborators on the of localizations at infinity for C^*-sub-algebras of $\mathcal{C}^b_u(X)$. A prominent role here is played by representations induced from characters. In Section 3, we introduce the basic algebras $\mathcal{E}_\mathcal{S}(X)$ generated by functions of the form $v_Y \circ \pi_Y$ and study radial limits at infinity for functions in these algebras. In order to describe the (character) spectrum of $\mathcal{E}_\mathcal{S}(X)$ *as a set*, we introduce the concepts of an "$\mathcal{S}$-chain" and of an "augmented $\mathcal{S}$-chain." An $\mathcal{S}$*-flag* is a sequence $\mathcal{Z}$ of distinct subspaces $0 = Z_0 \subset Z_1 \subset \ldots \subset Z_k$ in $\mathcal{S}$. An $\mathcal{S}$*-chain* associated to the $\mathcal{S}$-flag $\mathcal{Z}$ is a sequence $\overrightarrow{\alpha} := (\alpha_1, \alpha_2, \ldots, \alpha_k)$, with $\alpha_j \in \mathbb{S}_{Z_j/Z_{j-1}}$ (the sphere at infinity of Z_j/Z_{j-1}) such that Z_j is the least subspace in $\mathcal{S}$ containing α_j (in an obvious sense). An augmented $\mathcal{S}$-chain is a pair $(a, \overrightarrow{\alpha})$, where $\overrightarrow{\alpha}$ is an $\mathcal{S}$-chain associated to $\mathcal{Z} = (Z_0, Z_1, \ldots, Z_k)$ and $a \in X/Z_k$.

We then show that the set of characters of $\mathcal{E}_{\mathcal{S}}(X)$ is in a natural bijection with the set of augmented $\mathcal{S}$-chains (Theorem 3.9). In section 4 we obtain some results on the topology on the spectrum of $\mathcal{E}_{\mathcal{S}}(X)$. In Section 5, we use the result of Section 3 and 4 to give a description of the spectrum of Georgescu's algebra $\mathcal{G}_{\mathcal{S}}(X)$ introduced in his study of the N-body problem. Georgescu's algebra is generated by $\mathcal{C}_0(X/Y)$, and hence more natural for the study of the N-body problem. Its spectrum seems to be more singular, however. The topology on the spectrum of Georgescu's algebra $\mathcal{G}_{\mathcal{S}}(X)$ is studied by noticing that $\mathcal{G}_{\mathcal{S}}(X)$ is contained in $\mathcal{E}_{\mathcal{S}}(X)$. In the final section 6, we turn our attention to the crossed product algebra $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$. We then use result of [10, 35] to describe its primitive ideals space and to show that the family τ_α, $\alpha \in \mathbb{S}_X$, is an exhausting family of representations of $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$, which leads to the more precise result on the essential spectrum in Theorem 1.1.

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2. Crossed products and localizations at infinity

We now review some basic constructions and results. Most of them are due to Georgescu and its collaborators.

Let $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ denote the subalgebra of bounded uniformly continuous functions on X and let $\mathcal{C}_0(X)$ denote its ideal of functions vanishing at infinity. They act naturally on $L^2(X)$ by multiplication We also let $\mathcal{K}(X) := \mathcal{K}(L^2(X))$ be the ideal of compact operators on the same space.

Consider a commutative C^*-algebra $\mathcal{A}$ with (character) spectrum $\widehat{\mathcal{A}}$. It consists of the non-zero algebra morphisms $\chi : \mathcal{A} \to \mathbb{C}$ (all morphisms of C^*-algebras considered in this paper will be $*$-morphisms). If $\mathcal{A}$ is unital, then $\widehat{\mathcal{A}}$ is a compact topological space for the weak topology. In general, it is locally compact and the Gelfand transform $\Gamma_{\mathcal{A}} : \mathcal{A} \to \mathcal{C}_0(\widehat{\mathcal{A}})$, $\Gamma_{\mathcal{A}}(u)(\chi) := \chi(u)$, defines an isometric algebra isomorphism. If $\mathcal{A} \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ is invariant for the action of X, then X will act continuously on $\widehat{\mathcal{A}}$ and we shall denote by $\mathcal{A} \rtimes X$ the resulting crossed product algebra, see [24, 35]. Here the real vector space X is regarded as a locally compact, abelian group in the obvious way. Recall [12] that if $\mathcal{A}$ is a translation invariant C^*-subalgebra of $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$, then an isomorphic realization of the cross-product algebra $\mathcal{A} \rtimes X$ is the norm closed subalgebra of $\mathcal{B}(L^2(X))$ generated by the operators of the form $u(q)v(p)$, where $u \in \mathcal{A}$ and $v \in \mathcal{C}_0(X^*)$. This is an important feature that we now pause to briefly discuss.

More precisely, we have a natural morphism $\pi_0 : \mathcal{A} \rtimes X \to \mathcal{B}(L^2(X))$ obtained from the canonical actions of $\mathcal{A}$ and X on $L^2(X)$, since they form a covariant representation of $(\mathcal{A}, X, \tau)$ [10, 35]. Let us call this representation the *spatial representation* of $\mathcal{A} \rtimes X$. The result in [12] is that π_0 is injective. Indeed, since X is amenable, we can consider the reduced cross-product $\mathcal{A} \rtimes_r X \simeq \mathcal{A} \rtimes X$, which is defined as the completion of the algebra generated by $\tilde{m}_f h(p)$ acting on $\mathcal{B}(L^2(X) \otimes L^2(X))$ (two copies!), where $\tilde{m}_f$, $f \in \mathcal{A}$, acts on $L^2(X) \otimes L^2(X) \simeq L^2(X \times X)$ as the multiplication with the function $(x, y) \to f(x - y)$ and $h(p)$ is the convolution operator in the second variable X of $X \times X$. Given the special feature of this construction, we can rearrange our action, up to an isomorphism, to become independent of the first variable, and hence simply a large multiple of the spatial representation π_0, which is hence also injective, and therefore an isometry onto its image. For simplicity, we shall identify the abstract algebra $\mathcal{A} \rtimes X$ with its (isometric) image trough π_0 in $\mathcal{B}(L^2(X))$.

We shall need some more specific consequences for the algebra $\mathcal{A} \rtimes X$. Recall that an ideal of a C^*-algebra B is called *primitive* if it is the kernel of an irreducible representation. Then the *primitive ideal spectrum* $\operatorname{Prim}(B)$ of B is the set of primitive ideals of B [9]. For each closed two-sided ideal I of B, we denote by $\operatorname{Prim}_I(B)$ the set of all primitive ideals of B containing I. The sets of the form $\operatorname{Prim}_I(B)$ are the closed subsets in a topology on $\operatorname{Prim}(B)$, called the *Jacobson topology*. If B is commutative, then $\operatorname{Prim}(B) \cong \widehat{B}$ are naturally homeomorphic, so we may occasionally identify these spaces in what follows.

Definition 2.1. *A two sided ideal J of A is* essential *in A if $aJ = 0$ implies $a = 0$.*

Let us assume from now on that $\mathcal{C}_0(X) \subset \mathcal{A}$. Then $\pi_0(\mathcal{C}_0(X) \rtimes X)$ consists of the ideal $\mathcal{K}(X)$ of compact operators on $L^2(X)$. In particular, the spatial representation π_0 is actually an irreducible representation $\pi_0 : \mathcal{A} \rtimes X \to \mathcal{B}(L^2(X))$. A consequence of the injectivity of the spatial representation π_0 is that

Lemma 2.2. *The ideal $\mathcal{C}_0(X) \rtimes X \subset \mathcal{A} \rtimes X$ is an essential ideal of $\mathcal{A} \rtimes X$.*

Proof. Let $a \in \mathcal{A} \rtimes X$ be such that $a\mathcal{C}_0(X) \rtimes X = 0$. We notice that if $T \in \mathcal{B}(L^2(X))$ is a bounded operator such that $T\mathcal{K}(X) = 0$, then $T = 0$. Then we use this observation for $T = \pi_0(a)$ conclude that $\pi_0(a) = 0$ and hence $a = 0$ by the injectivity of π_0. $\square$

We shall need the following remark in the last section.

Remark 2.3. Since the vector representation $\pi_0 : \mathcal{A} \rtimes X \to \mathcal{B}(L^2(X))$ is irreducible, the zero ideal, that is, the kernel of π_0, is a primitive ideal of $\mathcal{A} \rtimes X$. We let 0 denote the zero ideal, for simplicity. Then $0 \in \operatorname{Prim}(\mathcal{A} \rtimes X)$. Moreover, $\{0\}$ is also an open subset of $\operatorname{Prim}(\mathcal{A} \rtimes X)$ that corresponds to the ideal $\mathcal{K}(X)$ of compact operators on $L^2(X)$.

Let τ_a the action of $a \in X$ by translations on our algebras of functions. If P is an operator on $L^2(X)$, then its translation by $x \in X$ is defined by the relation $\tau_x(P) := T_x^* P T_x$, as in the introduction.

Consider a character $\chi \in \widehat{\mathcal{A}}$ and define, for $u \in \mathcal{A}$, the function $\tau_\chi(u) : X \to \mathbb{C}$ by

$$\tau_\chi(u)(y) := \chi(\tau_y(u)) . \tag{9}$$

Then $\tau_\chi : \mathcal{A} \to \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ and, if we denote by $\chi_x : \mathcal{A} \to \mathbb{C}$ the evaluation at x, then $\tau_{\chi_x} = \tau_x$, as is seen from the relation $\tau_x(u)(y) = u(x+y) = \chi_x(\tau_y(u))$. We denote by $\tau_\chi \rtimes X : \mathcal{A} \rtimes X \to \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X$ the induced morphism. We have then the following basic result from [13].

Theorem 2.4. *Assume that $\mathcal{C}_0(X) \subset \mathcal{A} \subset \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$. Then the induced morphism*

$$\prod_{\chi \in \widehat{\mathcal{A}} \smallsetminus X} \tau_\chi \rtimes X \,:\, \mathcal{A} \rtimes X \longrightarrow \prod_{\chi \in \widehat{\mathcal{A}} \smallsetminus X} \mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X) \rtimes X. \tag{10}$$

has kernel $\mathcal{K}(X)$, the ideal of compact operators on $L^2(X)$.

In particular, an operator $P \in \mathcal{A} \rtimes X$ is compact if, and only if, $\tau_\chi \rtimes X(P) = 0$, for all $\chi \in \widehat{\mathcal{A}} \smallsetminus X$. Here we have used the fact that every character of a closed, two-sided ideal of a C^*-algebra extends *uniquely* to the algebra. In particular, we have that $X \cong \widehat{\mathcal{C}_0(X)} \subset \widehat{\mathcal{A}}$. This explains the notation $\widehat{\mathcal{A}} \smallsetminus X$. This theorem gives right away the following corollary. For any P, we define its *essential spectrum* $\sigma_{\mathrm{ess}}(P)$ as the set of those $\lambda \in \mathbb{C}$ such that $P - \lambda$ is *not* Fredholm. In case P is unbounded, we regard it as a bounded operator on its domain endowed with the graph norm.

Corollary 2.5. *If P is in $\mathcal{A} \rtimes X$ or is affiliated to it, then $\sigma_{\mathrm{ess}}(P) = \overline{\cup}_{\chi \in \widehat{\mathcal{A}} \smallsetminus X} \sigma(\tau_\chi(P))$.*

To successfully use these results, we thus need to identify the spectrum $\widehat{\mathcal{A}}$ of $\mathcal{A}$. It would be interesting to see the relevance of these results for a questions raised by Exel on the properties of regular representations for groupoids [11].

3. Character spectrum and chains of subspaces

In this section, we determine the spectrum of $\mathcal{E}_{\mathcal{S}}(X)$ *as a set*. The topology will be discussed in the next section.

Recall that in this paper $\mathcal{S}$ denotes a (non empty) semi-lattice of sub-spaces of X, that is, $Z_1 \cap Z_2 \in \mathcal{S}$ if $Z_1, Z_2 \in \mathcal{S}$. If $X \notin \mathcal{S}$, then $\mathcal{S}' = \mathcal{S} \cup \{X\}$ is a semi-lattice of subspaces of X with $\mathcal{E}_{\mathcal{S}'}(X) = \mathcal{E}_{\mathcal{S}}(X)$. There is thus no loss of generality to assume that $X \in \mathcal{S}$, which we shall do from now on.

Remark 3.1. The algebras $\mathcal{E}_{\mathcal{S}}(X)$ make sense for any non empty family $\mathcal{S}$ of sub-spaces of X. It is convenient however for us to assume that $\mathcal{S}$ is a semi-lattice since then $\mathcal{S}$ has a least element Y_0 and then $\mathcal{E}_{\mathcal{S}}(X)$ is isomorphic to $\mathcal{E}_{\mathcal{S}'}(X/Y_0)$, where $\mathcal{S}'$ is the induced semi-lattice on X/Y_0. We have $0 \in \mathcal{S}'$, which may not be the case for $\mathcal{S}$. Also note that $\mathcal{C}_0(X) \subset \mathcal{E}_{\mathcal{S}}(X)$ if, and only if, $0 \in \mathcal{S}$. In order to apply the results of Section 2, we thus need to assume that $0 \in \mathcal{S}$. In the important example of the semi-lattice $\mathcal{S}_N$ mentioned in the Introduction, we do have that $0 \in \mathcal{S}_N$, but that is not true for the semi-lattice generated just by the subspaces $\mathcal{P}_{ij}$. If $0 \notin \mathcal{S}$, then H is among the operators $\tau_\alpha(H)$, so Theorem 1.1 simply asserts that $\sigma_{\mathrm{ess}}(H) = \sigma(H)$, which is clear anyway, since H is invariant with respect to the minimal element of the semi-lattice $\mathcal{S}$, which is non-zero if $0 \notin \mathcal{S}$.

3.1. **Translation to infinity.** The natural projection $\pi_Y : X \to X/Y$ extends by continuity to a map $\tilde{\pi}_Y : \overline{X} \smallsetminus \mathbb{S}_Y \to \overline{X/Y}$ satisfying $\tilde{\pi}_Y(\mathbb{S}_X \smallsetminus \mathbb{S}_Y) \subset \mathbb{S}_{X/Y}$. More precisely, if $\alpha \in \mathbb{S}_X \smallsetminus \mathbb{S}_Y$, then it is a half-line $\mathbb{R}_+ a$ in X, with $a \in X \smallsetminus Y$. Then $\tilde{\pi}_Y(\alpha)$ correspond at the half-line $\mathbb{R}_+^* \pi_Y(a)$ in X/Y. We note, however, that π_Y will not have a limit at $\alpha \in \mathbb{S}_Y$. Indeed, for each vector in $y \in \overline{X/Y}$, we can find a sequence $(x_n) \in X$ such that $\lim_{n \to +\infty} x_n = \alpha$ and $\lim_{n \to +\infty} \pi_Y(x_n) = y$.

Let $\alpha = \hat{a} \in \mathbb{S}_X$ (so $a \neq 0$). As in [14], if $u \in \mathcal{C}(\overline{X/Y})$, $x \in X$, then

$$\tau_\alpha(u)(x) := \lim_{r \to +\infty} u(ra + x) = \begin{cases} u(x) & \text{if } \alpha \subset Y \text{ (i.e., } a \in Y) \\ u(\tilde{\pi}_Y(\alpha)) \in \mathbb{C} & \text{otherwise} \end{cases} \tag{11}$$

exists, and hence the limit $\tau_\alpha(u)$ exists for all $u \in \mathcal{E}_{all}(X)$ (the algebra obtained by considering the case of *all* subspaces of X, as in [14]). In particular, we have that $\tau_\alpha(u) \in \mathcal{E}_{\mathcal{S}}(X)$, if $u \in \mathcal{E}_{\mathcal{S}}(X)$, and hence τ_α defines an endomorphism of the algebra $\mathcal{E}_{\mathcal{S}}(X)$. Note that the limit defining τ_α is both in pointwise sense for functions and in strong sense for operators on $L^2(X)$.

For $\alpha \in \mathbb{S}_X$, we shall denote by $\chi_\alpha(f) := f(\alpha)$, the evaluation character at α for $f \in \mathcal{C}(\overline{X})$. We have the following lemma [14]

Lemma 3.2. *Let $Y \subset X$ be a subspace, let B be the C^*–algebra generated by $\mathcal{C}(\overline{X})$ and $\mathcal{C}(\overline{X/Y})$ in $\mathcal{C}_u^b(X)$, and let $\alpha \in \mathbb{S}_X \smallsetminus \mathbb{S}_Y$. Then the character χ_α of $\mathcal{C}(\overline{X})$ extends to a unique character of B. This extension is the restriction of τ_α to B.*

We shall need the following notation. Let $\alpha \in \mathbb{S}_X$ and

$$\mathcal{S}_\alpha := \{Y \in \mathcal{S} |\, \alpha \subset Y\}\,, \quad Z(\alpha) := \bigcap_{Y \in \mathcal{S}_\alpha} Y\,, \quad \mathcal{S}/\alpha := \{Y/Z(\alpha) |\, Y \in \mathcal{S}_\alpha\}\,. \tag{12}$$

Then $\mathcal{S}_\alpha$ is again a semi-lattice. Therefore $Z(\alpha) \in \mathcal{S}_\alpha$ since $\dim(X) < \infty$, and hence it is the smallest element of $\mathcal{S}_\alpha$. Similarly, $\mathcal{S}/\alpha$ is the induced semi-lattice of subspaces of $X/Z(\alpha)$.

The semi-lattices $\mathcal{S}_\alpha$ and $\mathcal{S}/\alpha$ will play a fundamental role in what follows. For instance

$$\tau_\alpha(\mathcal{E}_{\mathcal{S}}(X)) \;=\; \mathcal{E}_{\mathcal{S}_\alpha}(X) \tag{13}$$

and $\mathcal{E}_{\mathcal{S}_\alpha}(X)$ is naturally isomorphic to $\mathcal{E}_{\mathcal{S}/\alpha}(X/Z(\alpha))$ via $\pi_{Z(\alpha)} : X \to X/Z(\alpha)$. We note that, unlike in the case of all sub-spaces of X, the semi-lattices $\mathcal{S}_\alpha$ and $\mathcal{S}/\alpha$ depend on $\mathcal{S}$, and not just on $\alpha \in \mathbb{S}_X$. The semi-lattice $\mathcal{S}_\alpha$ has the useful property that $0 \in \mathcal{S}_\alpha$.

Lemma 3.3. *The morphism τ_α descends to a surjective morphism*

$$\tilde{\tau}_\alpha : \mathcal{E}_{\mathcal{S}}(\,)X \to \; \mathcal{E}_{\mathcal{S}/\alpha}(X/Z(\alpha)) \;.$$

Let $\alpha \in \mathbb{S}_X$, regarded as a half line in X. Let $Z(\alpha)$ be the smallest subspace in $\mathcal{S}$ containing α, as before. Also, let $X' := X/Z(\alpha)$ and $\mathcal{S}' := \mathcal{S}/\alpha$. (Recall that $\mathcal{S}/\alpha := \{Y/Z(\alpha) \subset X' |\, Z(\alpha) \subset Y \in \mathcal{S}\}$.) Then we consider

$$\tau_\alpha^* \;:\; \widehat{\mathcal{E}_{\mathcal{S}'}(X')} \;\cong\; \operatorname{Prim}(\mathcal{E}_{\mathcal{S}/\alpha}(X/Z(\alpha))) \;\to\; \widehat{\mathcal{E}_{\mathcal{S}}(X)}\;, \tag{14}$$

the map dual to $\tilde{\tau}_\alpha$, that is, $\tau_\alpha^*(\chi) := \chi \circ \tilde{\tau}_\alpha$. The above lemma gives that τ_α^* is continuous and a homeomorphism onto its image, which is a closed, compact subset of $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$. The following lemma identifies the image of τ_α^* with the set of characters of $\mathcal{E}_{\mathcal{S}}(X)$ that restrict to χ_α on $\mathcal{C}(\overline{X})$ when $\mathcal{C}(\overline{X}) \subset \mathcal{E}_{\mathcal{S}}(X)$, that is when $0 \in \mathcal{S}$. In view of Remark 3.1, *we assume from now that* $0 \in \mathcal{S}$.

Lemma 3.4. *Let $\alpha \in \mathbb{S}_X$ and $\Omega_\alpha := \{\chi \in \widehat{\mathcal{E}_{\mathcal{S}}(X)} |\, \chi|_{\mathcal{C}(\overline{X})} = \chi_\alpha\}$ (recall that $0 \in \mathcal{S}$). Then*

$$\Omega_\alpha \;=\; Im(\tau_\alpha^*) \;\cong\; \operatorname{Prim}(\mathcal{E}_{\mathcal{S}/\alpha}(X/Z(\alpha)))\;.$$

In other words, we have that a character $\chi \in \widehat{\mathcal{E}_{\mathcal{S}}(X)}$ restricts to the character χ_α on $\mathcal{C}(\overline{X})$ if, and only if, it is of the form $\chi = \chi' \circ \tilde{\tau}_\alpha$, for some character χ' of $\mathcal{E}_{\mathcal{S}/\alpha}(X/Z(\alpha))$.

Conversely, given a character χ of $\mathcal{E}_{\mathcal{S}}(X)$, let us consider its restriction to a character of $\mathcal{C}(\overline{X}) \subset \mathcal{E}_{\mathcal{S}}(X)$. Hence there exists $\alpha \in \overline{X}$ such that $\chi = \chi_\alpha$ on $\mathcal{C}(\overline{X})$. If $\alpha \in X \subset \overline{X}$, then, in fact, χ is uniquely determined by α, since $X \cong \widehat{\mathcal{C}_0(X)}$ and every character of an ideal extends *uniquely* to the algebra. In particular, we obtain that X identifies with an open subset of $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$. We shall write $X \subset \widehat{\mathcal{E}_{\mathcal{S}}(X)}$, by abuse of notation. If $\alpha \notin X$, we have that $\alpha \in \mathbb{S}_X$, and hence $\chi \in \Omega_\alpha \cong \operatorname{Prim}(\mathcal{E}_{\mathcal{S}/\alpha}(X/Z(\alpha)))$.

Lemma 3.5. *Assume $0 \in \mathcal{S}$, as before. The restriction map $R : \widehat{\mathcal{E}_{\mathcal{S}}(X)} \to \overline{X}$ associated to the inclusion $\mathcal{C}(\overline{X}) \subset \mathcal{E}_{\mathcal{S}}(X)$ gives rise to a disjoint union decomposition*

$$\widehat{\mathcal{E}_{\mathcal{S}}(X)} \;=\; R^{-1}(X) \cup_{\alpha \in \mathbb{S}_X} R^{-1}(\{\alpha\}) \;=:\; X \cup_{\alpha \in \mathbb{S}_X} \Omega_\alpha\;.$$

This allows us an inductive determination of the spectrum of $\mathcal{E}_{\mathcal{S}}(X)$ since Ω_α identifies with the spectrum of $\mathcal{E}_{\mathcal{S}/\alpha}(X/Z(\alpha))$. This inductive determination is conveniently formulated in terms of "chains," which we introduce next. We note that the subsets X and $\{\alpha\}$ in the above lemma are exactly the orbits of X acting on $\overline{X}$. A similar chain structure has appeared also in [2, 14].

3.2. $\mathcal{S}$**-chains.** The spectrum of the algebra $\mathcal{E}_{\mathcal{S}}(X)$ is conveniently described in terms of $\mathcal{S}$*-chains* $\overrightarrow{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_k)$, in a manner similar, but different to the one in [14]. To introduce the concept of $\mathcal{S}$-chains, we shall use the notation introduced in 12. An $\mathcal{S}$-chain $\overrightarrow{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_k)$, $0 \le k \le \dim(X)$, is required to satisfy the following recursive conditions, which involves also a sequence Z_j that is defined recursively as follows:

(1) $Z_0 = 0$;
(2) $\alpha_j \in \mathbb{S}_{X/Z_{j-1}}$, (a half-line in X/Z_{j-1}), $j = 1, 2, \dots, k$;
(3) $Z_j \in \mathcal{S}$ is the least subspace containing Z_{j-1} and α_j, for $j \le k$.

In (3) above, we have regarded $\alpha_j \in \mathbb{S}_{X/Z_{j-1}} \subset \overline{X/Z_{j-1}}$ as a half-line in X/Z_{j-1}, and hence, in turn, as a subset of X. It thus makes sense to ask whether α_j is a subset of Z_j or not, since Z_j is also a subset of X. In particular, we obtain $\alpha_1 \in \mathbb{S}_X$ and $Z_1 = Z(\alpha_1)$, that is, Z_1 is the least subspace of $\mathcal{S}$ containing α_1.

We shall say that the $\mathcal{S}$-chain $\overrightarrow{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_k)$ has *length* k. There is only one $\mathcal{S}$-chain of length zero: the empty set $\emptyset$.

The $\mathcal{S}$-chain $\overrightarrow{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_k)$ determines the spaces Z_j, $0 \le j \le k$, as follows. Let $\alpha'_j \in X$ be a representative of $\alpha_j \in \mathbb{S}_{X/Z_{j-1}}$. That is, $\alpha_j = \mathbb{R}_+^* \alpha'_j + Z_{j-1} \subset Z_j \in \mathcal{S}$. Let us fix $1 \le r \le k$. The subspace $[\alpha'_1, \alpha'_2, \dots, \alpha'_r] \subset X$ linearly generated by the vectors $\alpha'_1, \alpha'_2, \dots, \alpha'_r$, may depend on the choices of the α'_j, but the *least* subspace $Z \subset \mathcal{S}$ containing $[\alpha'_1, \alpha'_2, \dots, \alpha'_r]$ will *not* depend on the choices of the representatives α'_j and $Z_r = Z$. We shall thus occasionally also use the more complete notation

$$Z(\alpha_1, \alpha_2, \dots, \alpha_j) := Z_j \tag{15}$$

and $Z(\overrightarrow{\alpha}) := Z(\alpha_1, \alpha_2, \dots, \alpha_k)$ if $\overrightarrow{\alpha}$ has length k. If $\overrightarrow{\alpha} = \emptyset$ (that is, if $k = 0$), we let $Z(\overrightarrow{\alpha}) = 0$. The symbol $\tilde{\Xi}_X^{(k)}$ will denote the set of $\mathcal{S}$-chains of length k.

A sequence $0 \neq Z_1 \subsetneq Z_2 \subsetneq \dots \subsetneq Z_k$ of subspaces in $\mathcal{S}$ will be called an $\mathcal{S}$*-flag* (of length k). Each $\mathcal{S}$-flags of length k corresponds to at least one $\mathcal{S}$-chains of length k.

An *augmented* $\mathcal{S}$*-chain* is a pair $(a, \overrightarrow{\alpha})$, where $\overrightarrow{\alpha}$ is an $\mathcal{S}$-chain and $a \in X/Z(\overrightarrow{\alpha})$. By $\Xi_X^{(k)}$ we shall denote the set of augmented $\mathcal{S}$-chains of length k:

$$\Xi_X^{(k)} := \{(a, \overrightarrow{\alpha}) |\, \overrightarrow{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_k) \in \tilde{\Xi}_X^{(k)},\ a \in X/Z(\overrightarrow{\alpha})\} \,. \tag{16}$$

We let $\Xi_X := \cup_k \Xi_X^{(k)}$ denote the set of all augmented $\mathcal{S}$-chains.

Assume $(a, \overrightarrow{\alpha}) = (a, \alpha_1, \alpha_2, \dots, \alpha_k) \in \Xi_X$ and let $\mathcal{S}_j := \{Y/Z_j |\, Z_j \subset Y, Y \in \mathcal{S}\}$ be the induced semi-lattice of subspaces of X/Z_j, as before, see Equation (15). We obtain for each j (so $\alpha_j \in X/Z_{j-1}$) a morphism

$$\tilde{\tau}_{\alpha_j} : \mathcal{E}_{\mathcal{S}_{j-1}}(X/Z_{j-1}) \ \to\ \mathcal{E}_{\mathcal{S}_j}(X/Z_j) \,. \tag{17}$$

Recall that if $a \in X$, the character $\chi_a : \mathcal{E}_{\mathcal{S}}(X) \to \mathbb{C}$ is the evaluation at a.

Definition 3.6. *For each augmented* $\mathcal{S}$*-chain* $(a, \overrightarrow{\alpha}) := (a, \alpha_1, \alpha_2, \dots, \alpha_k) \in \Xi_X^{(k)}$*, we define*

$$\tau_{\overrightarrow{\alpha}} \ :=\ \tilde{\tau}_{\alpha_k} \circ \tilde{\tau}_{\alpha_{k-1}} \circ \dots \circ \tilde{\tau}_{\alpha_1} \ :\ \mathcal{E}_{\mathcal{S}}(X) \ \to\ \mathcal{E}_{\mathcal{S}_k}(X/Z_k) \ =\ \mathcal{E}_{\mathcal{S}/\overrightarrow{\alpha}}(X/Z(\overrightarrow{\alpha}))$$

and $\chi_{a,\overrightarrow{\alpha}} \ :=\ \chi_a \circ \tau_{\overrightarrow{\alpha}} : \mathcal{E}_{\mathcal{S}}(X) \to \mathbb{C}$.

Since the map $\tau_{\overrightarrow{\alpha}}$ of Definition 3.6 is a composition of C^*-algebra morphisms, it is a C^*-algebra morphism itself, and hence $\chi_{a,\overrightarrow{\alpha}}$ defines a character of $\mathcal{E}_{\mathcal{S}}(X)$.

Lemma 3.7. *The composite morphism* $\tau_{\overrightarrow{\alpha}} : \mathcal{E}_{\mathcal{S}}(X) \to \mathcal{E}_{\mathcal{S}/\overrightarrow{\alpha}}(X/Z(\overrightarrow{\alpha}))$ *of Definition 3.6 is surjective.*

It will be convenient to use also the more complete notation

$$\mathcal{S}/(\alpha_1, \alpha_2, \dots, \alpha_j) := \{ Y/Z(\alpha_1, \alpha_2, \dots \alpha_j) \mid [\alpha_1', \alpha_2' \dots \alpha_j'] \subset Y \in \mathcal{S} \} = \mathcal{S}_j .$$

Proof. The first assertion is a successive application of lemma 3.3. More precisely, we have the following sequence of surjective maps:

$$\mathcal{E}_{\mathcal{S}}(X) \xrightarrow{\tau_{\alpha_1}} \mathcal{E}_{\mathcal{S}/\alpha_1}(X/Z(\alpha_1)) \xrightarrow{\tau_{\alpha_2}} \mathcal{E}_{\mathcal{S}/(\alpha_1,\alpha_2)}(X/Z(\alpha_1,\alpha_2)) \xrightarrow{\tau_{\alpha_3}} \\ \dots \xrightarrow{\tau_{\alpha_k}} \mathcal{E}_{S/\overrightarrow{\alpha}}(X/Z(\overrightarrow{\alpha}))$$

Then the second assertion is direct consequence of the first one because, if $a \in X/Z(\overrightarrow{\alpha})$, then χ_a is a character of $\mathcal{E}_{S/\overrightarrow{\alpha}}(X/Z(\overrightarrow{\alpha}))$. □

Remark 3.8. *We distinguish two special cases :*

- *If $\overrightarrow{\alpha} = \emptyset$, we have $\tau_{\overrightarrow{\alpha}} = Id$, and hence $\chi_{a,\emptyset} := \chi_a$ ($a \in X$).*
- *If $Z(\overrightarrow{\alpha}) = X$, we have $\chi_{a,\overrightarrow{\alpha}} := \tau_{\overrightarrow{\alpha}}$, since there is only one $a \in X/X = 0$.*

We obtain that the spectrum of our algebra $\mathcal{E}_{\mathcal{S}}(X)$ identifies naturally with the set Ξ_X of augmented $\mathcal{S}$-chains.

Theorem 3.9. *Assume that $0 \in \mathcal{S}$, then we have a bijective map $\Theta : \Xi_X \to \widehat{\mathcal{E}_{\mathcal{S}}(X)}$,*

$$\Theta(a, \overrightarrow{\alpha}) := \chi_{a,\overrightarrow{\alpha}} := \chi_a \tau_{\overrightarrow{\alpha}} .$$

Proof. The proof is obtained by induction on $\dim(X)$, using Lemmas 3.3 and 3.4. □

Let us explain now how the characters $\chi_{a,\overrightarrow{\alpha}}$ act on $\mathcal{E}_{\mathcal{S}}(X)$. If $Z \subset Y \subset X$, we shall use the similar notation $\pi_{Y/Z} : X/Z \to X/Y$ for the linear projection, which we extend by continuity to $\tilde{\pi}_{Y/Z} : \overline{X/Z} \setminus \mathbb{S}_{Y/Z} \to \overline{X/Y}$.

Remark 3.10. Let $(a, \overrightarrow{\alpha}) \in \Xi_X$.

(1) If $\overrightarrow{\alpha} = \emptyset$, then $(a, \overrightarrow{\alpha}) = a$ and

$$\chi_{(a,\emptyset)}(f) = \chi_a(f) = f(a) .$$

(2) If $\overrightarrow{\alpha} \neq \emptyset$ has length $k \geq 1$ and $f \in \mathcal{C}(\overline{X/Y})$, with $Y \in \mathcal{S}$, we have

$$\chi_{(a,\overrightarrow{\alpha})}(f) = \begin{cases} f(\pi_{Y/Z(\overrightarrow{\alpha})}(a)) & \text{if } Z(\overrightarrow{\alpha}) \subset Y \\ f(\tilde{\pi}_{Y/Z_{p-1}}(\alpha_p)) & \text{if } Z_{p-1} \subset Y, \text{ but } Z_p \not\subset Y . \end{cases} \tag{18}$$

In the first case of Equation (18), $\pi_{Y/Z(\overrightarrow{\alpha})}(a) \in X/Y$ is well defined since $a \in X/Z(\overrightarrow{\alpha})$ and $Z(\overrightarrow{\alpha}) \subset Y$. In the second case, the index $0 < p \leq k$ is determined to be the largest satisfying $Z_{p-1} := Z(\alpha_1, \dots, \alpha_{p-1}) \subset Y$, (so $Z_p := Z(\alpha_1, \dots, \alpha_p) \not\subset Y$). This follows by repeatedly using Equation (11). We also notice that the relation $Z_p \not\subset Y$ is equivalent to $\alpha_p \notin Y/Z_{p-1}$. Again, $\tilde{\pi}_{Y/Z(\overrightarrow{\alpha})}(\alpha_p) \in \mathbb{S}_{X/Y}$ is defined since $\alpha_p \in \mathbb{S}_{X/Z_{p-1}}$, $Z_{p-1} \subset Y$, and $\alpha_p \notin \mathbb{S}_{Y/Z_{p-1}}$. See the definition of the extensions $\tilde{\pi}_Y$ at the beginning of this section.

From this remark it follows that the induced action of X on the set of augmented $\mathcal{S}$-chains Ξ_X is by translation on the first component:

$$x \cdot (a, \overrightarrow{\alpha}) = (\pi_{Z(\overrightarrow{\alpha})}(x) + a, \overrightarrow{\alpha}) , \quad x \in X, \text{ and } (a, \overrightarrow{\alpha}) \in \Xi_X . \tag{19}$$

In particular, if $Z(\overrightarrow{\alpha}) = X$, then $\overrightarrow{\alpha}$ is invariant for the action of X.

Remark 3.11. As in [14], if χ is the character of $\mathcal{E}_{\mathcal{S}}(X)$ associated to the augmented $\mathcal{S}$-chain $(a, \overrightarrow{\alpha})$, then $\tau_\chi(P) = \tau_a \tau_{\overrightarrow{\alpha}}(P)$ for $P \in \mathcal{E}_{\mathcal{S}}(X)$, and hence, also for $P \in \mathcal{E}_{\mathcal{S}}(X) \rtimes X$. This identifies all *limit operators* associated to P. The defintion of τ_χ from [13] was recalled in Section 2.

We would like next to study the topology on the space $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ of characters of $\mathcal{E}_{\mathcal{S}}(X)$ and the topology that it induces on $\Xi_X := \cup_{0 \le k \le \dim(X)} \Xi_X^{(k)}$, since this will be useful in proving that the family of morphisms $\{\tau_\alpha | \, \alpha \in \mathbb{S}_X\}$ is *exhaustive* (the notion of exhausting families was introduced in [23] and will be recalled in the last section (see Definition 6.3).

4. The topology on the spectrum of $\mathcal{E}_{\mathcal{S}}(X)$

We now give a first description of the *topology* on the spectrum of $\mathcal{E}_{\mathcal{S}}(X)$ by identifying it with a closed subset of the product $\prod_{Y \in \mathcal{S}} \overline{X/Y}$. We continue to assume in this section and thereafter, for simplicity, that $0, X \in \mathcal{S}$, even if some results hold in greater generality.

For each closed two-sided ideal I of A, we denote by $\mathrm{Prim}^I(A) := \mathrm{Prim}(A) \smallsetminus \mathrm{Prim}_I(A)$ the set of primitive ideals of A that *do not* contain I. (The sets of the form $\mathrm{Prim}^I(A)$ are thus the *open* subsets of $\mathrm{Prim}(A)$ in the the Jacobson topology). Recall the definition of an essential ideal (Definition 2.1). We have:

Proposition 4.1. *If J is an essential ideal of A then $Prim^J(A)$ is dense in $Prim(A)$.*

The converse is obviously true.

Remark 4.2. We shall use this result for $\mathcal{E}_{\mathcal{S}}(X)$ and $\mathcal{C}_0(X)$ and for their cross-products by X. In the first case, that is, for $A = \mathcal{E}_{\mathcal{S}}(X)$ and $J = \mathcal{C}_0(X)$, it follows from the definition that $\mathcal{C}_0(X)$ is essential in $\mathcal{E}_{\mathcal{S}}(X)$ (since it is essential in $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$), and hence that $X \cong \widehat{\mathcal{C}_0(X)}$ (or rather that its image) is dense in $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$. In the second case, that is for $J := \mathcal{K}(X) \cong \mathcal{C}_0(X) \rtimes X \subset \mathcal{E}_{\mathcal{S}}(X) \rtimes X =: A$, we have already seen in Remark 2.3 that $\mathrm{Prim}^J = \{0\}$. Indeed, this follows by taking $\mathcal{A} := \mathcal{E}_{\mathcal{S}}(X)$ in that remark. It thus follows that 0 is a dense point in the primitive ideal spectrum of $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$: $\overline{\{0\}} = \mathrm{Prim}(\mathcal{A} \rtimes X)$.

Let us combine all projections $\pi_Y : X \to X/Y$ into $G^{\mathcal{S}}(x) := (\pi_Y(x))_{Y \in \mathcal{S}}$,

$$G^{\mathcal{S}} \ := \prod_{Y \in \mathcal{S}} \pi_Y \ : \ X \ \to \ \prod_{Y \in \mathcal{S}} \overline{X/Y} \, . \tag{20}$$

Let us similarly consider all the restrictions $\widehat{\mathcal{E}_{\mathcal{S}}(X)} \to \widehat{\mathcal{C}(\overline{X/Y})} \cong \overline{X/Y}$. Combining all these restrictions, we obtain the map $\Phi : \widehat{\mathcal{E}_{\mathcal{S}}(X)} \to \prod\limits_{Y \in \mathcal{S}} \overline{X/Y}$

$$\Phi(\chi) \ = \ (x_Y) \ \in \ \prod_{Y \in \mathcal{S}} \overline{X/Y} \, , \quad \text{where } \ \chi(f) = f(x_Y) \, , \ \ f \in \mathcal{C}(\overline{X/Y}), \ Y \in \mathcal{S} \, . \tag{21}$$

Lemma 4.3. *The map Φ of Equation* (21) *is continuous and a homeomorphism onto its image.*

Proof. The continuity of Φ is due to the fact that the dual map defined by restriction for characters is continuous. The injectivity comes from the fact that the algebras $\mathcal{C}(\overline{X/Y})$ generate $\mathcal{E}_{\mathcal{S}}(X)$. The proof is completed by recalling that a continuous bijection of compact spaces is a homeomorphism. □

See [20, 25] and the references therein for more results of this type and other applications. Let $j : X \to \widehat{\mathcal{E}_{\mathcal{S}}(X)}$ be the inclusion defined by $\mathcal{C}_0(X) \subset \mathcal{E}_{\mathcal{S}}(X)$. Also, recall the map Φ defined in Equation (21) and $G^{\mathcal{S}}$ defined in Equation (20). The following theorem describes the topology on $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$.

Theorem 4.4. *The following diagram is commutative*

$$
\begin{tikzcd}
\widehat{\mathcal{E}_{\mathcal{S}}(X)} \arrow[rr, "\Phi"] & & \prod_{Y\in\mathcal{S}} \overline{X/Y} \\
& X \arrow[ul, "j"] \arrow[ur, "G^{\mathcal{S}}"'] &
\end{tikzcd}
\tag{22}
$$

In particular, Φ induces a homeomorphism of $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ onto $\overline{G_{\mathcal{S}}(X)}$ that is functorial in $\mathcal{S}$.

Proof. Each component of the composition $\Phi \circ j$ is obtained by extending a character χ_x of $\mathcal{C}_0(X/Y)$ to $\mathcal{E}_{\mathcal{S}}(X)$ and then restricting to $\mathcal{C}(\overline{X/Y})$. This extension is unique and correspoinds to the evaluation at x, that is, to χ_x. Since $\mathcal{C}_0(X)$ is an essential ideal in $\mathcal{E}_{\mathcal{S}}(X)$, X is dense in $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$. By continuity

$$\Phi(\widehat{\mathcal{E}_{\mathcal{S}}(X)}) \ \subset\ \overline{\Phi(j(X))} \ =\ \overline{G^{\mathcal{S}}(X)}\ .$$

Moreover, the image contains X and is closed, since it is compact. Hence we have equality. The result then follows from Lemma 4.3. □

The functoriality in $\mathcal{S}$ refers to the inclusion $\mathcal{E}_{\mathcal{S}}(X) \subset \mathcal{E}_{\mathcal{S}'}(X)$ if $\mathcal{S} \subset \mathcal{S}'$.

The meaning of Theorem 4.4 is that it provides also an elementary geometric construction of the space $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$, which, as we have already mentioned, may be useful for numerical methods. The description of the topology on $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ is, however, not completely satisfactory at this point, since we do not have a good understanding of $\overline{G_{\mathcal{S}}(X)}$ yet. We have good reasons to believe, however, that it is a manifold with corners obtained by successively blowing-up the singular strata and that it coincides with a space introduced by Vasy [34].

A natural question then is to identify the composite map $\Phi\circ\Theta : \Xi_X \to \overline{G_{\mathcal{S}}(X)}$. Recall that $\pi_{Y/Z} : X/Z \to X/Y$ is, as usual, the projection, and that it extends to a continuous map $\tilde{\pi}_{Y/Z} : \overline{X/Z} \smallsetminus \mathbb{S}_{Y/Z} \to \overline{X/Y}$. Given that $\Phi : \widehat{\mathcal{E}_{\mathcal{S}}(X)} \to \prod_{Y\in\mathcal{S}} \overline{X/Y}$ is defined by restrictions to the generating subalgebras $\mathcal{C}(\overline{X/Y})$, see (21), Remark 3.10 tells us that the Y component $\big(\Phi(\chi_{(a,\overrightarrow{\alpha})})\big)_Y \in \overline{X/Y}$ of $\Phi(\chi_{(a,\overrightarrow{\alpha})}) \in \prod_{Y\in\mathcal{S}}\overline{X/Y}$ is

$$\big(\Phi(\chi_{(a,\overrightarrow{\alpha})})\big)_Y \ =\ \begin{cases} \pi_{Y/Z(\overrightarrow{\alpha})}(a) & \text{if } Z(\overrightarrow{\alpha}) \subset Y \\ \tilde{\pi}_{Y/Z_{p-1}}(\alpha_p) & \text{if } Z_{p-1}\subset Y, \text{ but } Z_p \not\subset Y\ , \end{cases} \tag{23}$$

where we have used the notation of that remark. Let $\overrightarrow{\alpha} = (\alpha_1,\dots,\alpha_k)$. We note that the component of $\Phi(\chi_{(a,\overrightarrow{\alpha})})$ corresponding to $Y = Z_j$, $j = 0,\dots,k-1$, is α_{j+1}, whereas the component of $\Phi(\chi_{(a,\overrightarrow{\alpha})})$ corresponding to $Y = Z_k$ is a. Thus all other components of $\Phi(\chi_{(a,\overrightarrow{\alpha})}) = \Phi(\Theta(a,\overrightarrow{\alpha}))$ are determined by these components (a and α_j), as explained. More precisely, to determine the $Y \in \mathcal{S}$ component of $\Phi(\chi_{(a,\overrightarrow{\alpha})})$, we need to find the largest p such that $Z_{p-1} \subset Y$, and then the component corresponding to Y will be the projection onto $\overline{X/Y}$ of α_p, if $p < k$, or of a, if $p = k$.

Let us consider the augmented $\mathcal{S}$-chains $(a,\overrightarrow{\alpha}) \in \Xi_X^{(k)}$ that have the same fixed $\mathcal{S}$-flag $\mathcal{Z} := (Z_1, Z_2, ..., Z_k)$, where $Z_j := Z(\alpha_1,\alpha_2,\dots,\alpha_j) \in \mathcal{S}$, as before, and hence

$Z_1 \subset Z_2 \subset \ldots \subset Z_k$. If $\overrightarrow{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, then $\alpha_1 \in \mathcal{Y}_1 := \mathbb{S}_{Z_1} \smallsetminus \cup_{Y \in \mathcal{S}, Y \subsetneq Z_1} \mathbb{S}_Y$, and this set has a natural smooth structure and hence a natural topology. Similarly,

$$\alpha_j \in \mathcal{Y}_j := \mathbb{S}_{Z_j/Z_{j-1}} \smallsetminus \cup_{Y \in \mathcal{S}, Z_{j-1} \subset Y \subsetneq Z_j} \mathbb{S}_{Y/Z_{j-1}} \,,$$

and hence we can endow the set of $\mathcal{S}$-chains with the given flag $\mathcal{Z}$ with the induced topology of the product manifold $\mathcal{Y}_1 \times \mathcal{Y}_2 \times \ldots \times \mathcal{Y}_k$ and the set of augmented $\mathcal{S}$-chains with the given flag $\mathcal{Z}$ with the induced topology of the product manifold

$$\mathcal{X}_{\mathcal{Z}} \; := \; X/Z(\overrightarrow{\alpha}) \times \mathcal{Y}_1 \times \mathcal{Y}_2 \times \ldots \times \mathcal{Y}_k \,. \tag{24}$$

We then see that $\Phi \circ \Theta$ restricts to a diffeomorphism from $\mathcal{X}_{\mathcal{Z}}$ onto its image in $\prod_{Y \in \mathcal{S}} \overline{X/Y}$. Indeed, it is enough to consider the components of $\Phi \circ \Theta(a, \overrightarrow{\alpha})$ corresponding to all Z_j, $j = 0, \ldots, k$ (with $\mathcal{Y}_j$ projecting onto $\overline{X/Z_{j-1}}$). Clearly, all the sets $\Phi \circ \Theta(\mathcal{X}_{\mathcal{Z}})$ are disjoint and $\overline{G_{\mathcal{S}}(X)} = \cup_{\mathcal{Z}} \Phi \circ \Theta(\mathcal{X}_{\mathcal{Z}})$, since to each augmented $\mathcal{S}$-chain there corresponds exactly one $\mathcal{S}$-flag.

We endow the set of $\mathcal{S}$-flags with the lexicographic order. Namely, let us consider the $\mathcal{S}$-flags $\mathcal{Z} := (Z_1, Z_2, \ldots, Z_k)$ and $\mathcal{Z}' := (Z_1', Z_2', \ldots, Z_n')$. Then

$$\mathcal{Z} \,<\, \mathcal{Z}' \;\Leftrightarrow\; Z_1 = Z_1' \,,\; Z_2 = Z_2' \,,\; \ldots \,,\; Z_{j-1} = Z_{j-1}' \,,\;\; \text{and} \;\; Z_j \supsetneq Z_j' \,, \tag{25}$$

for some $j \leq \min\{k, n\}$. Clearly, if $\mathcal{Z}' < \mathcal{Z}''$ and $\mathcal{Z} < \mathcal{Z}'$, then $\mathcal{Z} < \mathcal{Z}''$. We can now look at the relation between the sets $\Phi \circ \Theta(\mathcal{X}_{\mathcal{Z}})$.

Lemma 4.5. *Let $\mathcal{Z}$ and $\mathcal{Z}'$ be two $\mathcal{S}$-flags such that $\Phi \circ \Theta(\mathcal{X}_{\mathcal{Z}'})$ intersects the closure of $\Phi \circ \Theta(\mathcal{X}_{\mathcal{Z}})$. Then $\mathcal{Z} < \mathcal{Z}'$.*

Proof. Let $j \geq 1$ be the smallest integer such that $Z_0 = Z_0'$, $Z_1 = Z_1'$, $Z_2 = Z_2'$, $\ldots$, $Z_{j-1} = Z_{j-1}'$, but $Z_j' \neq Z_j$. Let $(a', \overrightarrow{\alpha}') = (a', \alpha_1', \alpha_2', \ldots, \alpha_n')$ be an augmented $\mathcal{S}$-chain with flag $\mathcal{Z}'$ that maps to the closure of $\Phi \circ \Theta(\mathcal{X}_{\mathcal{Z}})$. Then each Y-component of $\Phi(\Theta(a', \overrightarrow{\alpha}'))$ is the limit of Y-components of points in $\mathcal{X}_{\mathcal{Z}}$, $Y \in \mathcal{S}$. This is true, in particular, for the Z_{j-1} component, which is in $\overline{X/Z_{j-1}}$. Then we see that $\alpha_j' \in \mathbb{S}_{Z_j/Z_{j-1}} \subset \overline{Z_j/Z_{j-1}}$, which gives $Z_j' \subset Z_j$, since Z_j' is the least subspace of $\mathcal{S}$ containing $Z_{j-1}' = Z_{j-1} \subset Z_j$ and α_j'. Hence $\mathcal{Z} < \mathcal{Z}'$, by definition. $\square$

Recall that a set in a topological space is *locally closed* if it is open in its closure, or, which is the same thing, if it is the intersection of an open subset and of a closed subset. We shall need the following corollary.

Corollary 4.6. *If $\mathcal{S}$ is finite, then for each $\mathcal{S}$-flag $\mathcal{Z}$, the set $\Phi \circ \Theta(\mathcal{X}_{\mathcal{Z}})$ is locally closed in $\overline{G_{\mathcal{S}}(X)}$.*

Proof. Let F be the union of all the sets $\Phi \circ \Theta(\mathcal{X}_{\mathcal{Z}'})$ with $\mathcal{Z} < \mathcal{Z}'$. Lemma 4.5 shows that the sets F and $F \cup \Phi \circ \Theta(\mathcal{X}_{\mathcal{Z}})$ are closed, since "$<$" is transitive. $\square$

5. GEORGESCU'S ALGEBRA

We now use the results of the previous subsection to obtain a quotient topology description of the topology on the spectrum of Georgescu's graded algebras [5].

Let us start from the same data: that is, we continue to assume that X is a finite dimension vector space and that $\mathcal{S}$ is a family of linear spaces of X such that $0, X \in \mathcal{S}$. In the framework of the *true* N-body problems, the interactions vanish at infinity, so it is more natural to consider the following algebra of interactions (potentials)

$$\mathcal{G}_{\mathcal{S}}(X) \; := \; \langle \mathcal{C}_0(X/Y) \rangle \,, \quad Y \in \mathcal{S} \,, \tag{26}$$

see [5] and the references therein. Notice that $X \in \mathcal{S}$ implies that $1 \in \mathcal{G}_{\mathcal{S}}(X)$.

As for the algebra $\mathcal{E}_{\mathcal{S}}(X)$, we want to describe the spectrum of the C^*-algebra $\mathcal{G}_{\mathcal{S}}(X)$. Since the natural map $\iota : \mathcal{G}_{\mathcal{S}}(X) \subset \mathcal{E}_{\mathcal{S}}(X)$ is an inclusion (injective), we have that the resulting dual map

$$\iota^* : \widehat{\mathcal{E}_{\mathcal{S}}(X)} \to \widehat{\mathcal{G}_{\mathcal{S}}(X)}\,, \quad \iota^*(\chi) := \chi_{|\mathcal{G}_{\mathcal{S}}(X)}\,, \tag{27}$$

is continuous and onto. As we already know explicitly $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$, it remains to determine the equivalence relation induced by ι^*.

Let $Y \in \mathcal{S}$ and $f \in \mathcal{C}_0(X/Y)$. Using Equation (18), we obtain

$$\chi_{(a,\overrightarrow{\alpha})}(\iota(f)) = \begin{cases} f(\pi_{Y/Z(\overrightarrow{\alpha})}(a)) & \text{if } Z(\overrightarrow{\alpha}) \subseteq Y \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

In summary the spectrum of the Georgescu algebra is then given by the following theorem.

Theorem 5.1. *Let $0, X \in \mathcal{S}$. The space $\widehat{\mathcal{G}_{\mathcal{S}}(X)}$ has the quotient topology for the map*

$$\iota^* : \widehat{\mathcal{E}_{\mathcal{S}}(X)} \to \widehat{\mathcal{G}_{\mathcal{S}}(X)}\,, \quad \iota^*(\chi) := \chi_{|\mathcal{G}_{\mathcal{S}}(X)}\,, \tag{29}$$

Moreover two characters $(a, \overrightarrow{\alpha})$ and $(b, \overrightarrow{\beta})$ in Ξ_X are equal on $\mathcal{G}_{\mathcal{S}}(X)$ if and only if $Z(\overrightarrow{\alpha}) = Z(\overrightarrow{\beta})$ and $a = b \in X/Z(\overrightarrow{\alpha}) = X/Z(\overrightarrow{\beta})$.

It is known that $\widehat{\mathcal{G}_{\mathcal{S}}(X)}$ is in a natural bijection with the disjoint union of the spaces X/Y, $Y \in \mathcal{S}$ [18]. The restriction map ι^* then becomes $\iota^*(a, \overrightarrow{\alpha}) = a \in X/Y$ for $Y = Z(\overrightarrow{\alpha})$. In particular, on each of the sets $\mathcal{X}_{\mathcal{Z}} := X/Z(\overrightarrow{\alpha}) \times \mathcal{Y}_1 \times \mathcal{Y}_2 \times \ldots \times \mathcal{Y}_k$ (see Equation 24), the map ι^* is simply the projection onto $X/Z(\overrightarrow{\alpha})$. This completely describes the topology on the quotient of this space as collapsing the "complicated part" $\mathcal{Y}_1 \times \mathcal{Y}_2 \times \ldots \times \mathcal{Y}_k$. We note, however, that several spaces of the form $\mathcal{X}_{\mathcal{Z}}$ may map to the same space $X/Z(\overrightarrow{\alpha})$, so further identifications may be in order.

Of course, the main point of our result is our belief that the space $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ has a topology that is less singular than that of $\widehat{\mathcal{G}_{\mathcal{S}}(X)}$. This seems to be justified by our preliminary results (see also [17, 18]) and we plan to pursue further this question in another publication. The concept from [20, 25] of "asymptotically independent" subalgebras of $\mathcal{C}_{\mathrm{b}}^{\mathrm{u}}(X)$ may be useful here.

6. Exhausting families and a precise result on the essential spectrum

In order to apply our results to Hamiltonians such as the one given in Equation (2), we need to study the cross-product C^*-algebra $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$. This C^*algebra is noncommutative, so we will consider exclusively its primitive ideal spectrum. We assume in this section that $\mathcal{S}$ is finite in order to use Corollary 4.6 and hence to be able to use the results in [35]. In particular, its spectrum $\widehat{\mathcal{E}_{\mathcal{S}}(X)} \cong \mathrm{Prim}(\mathcal{E}_{\mathcal{S}}(X))$ is second countable (i.e. it will have a countable basis of open subsets). Moreover, as we will see below, the action of X on $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ has locally closed orbits. This means that the primitive ideal spectrum of the cross-product C^*-algebra $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$ can be completely understood using, for instance, the theory explained in [10, 35].

More precisely, let us consider an arbitrary *locally compact, second countable space* Ω and assume that a *locally compact, second countable, abelian group* G, acts continuously on Ω. For simplicity, we shall assume that the orbits of G are locally closed in Ω, that is, that each orbit is open in its closure in Ω. The primitive ideal spectrum of $\mathcal{C}_0(\Omega) \rtimes G$ then consists of the set of pairs $(\mathcal{O}, \xi)$, where $\mathcal{O}$ is an orbit of G in Ω and ξ is a character of the stabilizer $G_{\mathcal{O}}$ of $\mathcal{O}$. (Recall that the stabilizer of the orbit $\mathcal{O} := G\omega$ is given by the set $G_\omega := \{g \in G |\ g\omega = \omega\}$, and this is independent of ω in the orbit $\mathcal{O}$, since G is commutative.) Moreover, the topology is the quotient topology of $\Omega \times \hat{G}$ with respect to the quotient map

$$\Psi_{\Omega,G} : \Omega \times \hat{G} \ \to \ \mathrm{Prim}(\mathcal{C}_0(\Omega) \rtimes G) ,$$

given by $\Psi_{\Omega,G}(\omega, \chi) := (G\omega, \chi|_{G_\omega})$, see Theorem 8.39 in [35] for details. This map is also natural with respect to restriction morphisms, in the following sense:

Proposition 6.1. *Assume that $\Omega' \subset \Omega$ is a closed, G-invariant subset, with Ω and G locally compact, second countable, as above. Then Ω' also has locally closed orbits and the inclusion $j : \Omega' \to \Omega$ induces a surjective morphism $j \rtimes G : \mathcal{C}_0(\Omega) \rtimes G \to \mathcal{C}_0(\Omega') \rtimes G$ and hence an injective map $(j \rtimes G)^* : \mathrm{Prim}(\mathcal{C}_0(\Omega') \rtimes G) \to \mathrm{Prim}(\mathcal{C}_0(\Omega) \rtimes G)$ such that*

$$(j \rtimes G)^* \circ \Psi_{\Omega',G} = \Psi_{\Omega,G} \circ (j \times id) \ : \ \Omega' \times \hat{G} \ \to \ \mathrm{Prim}(\mathcal{C}_0(\Omega) \rtimes G) .$$

A similar statement holds for open inclusions (but with the arrows reversed).

Proof. This follows from the fact that the stabilizer of $\omega \in \Omega'$ is the same as that of ω regarded as a point in Ω. See the proof of the Theorem 8.39 in [35]. □

We shall need the following corollary.

Corollary 6.2. *If the space Ω of Proposition 6.1 is a union $\Omega = \cup_{\alpha \in I} \Omega_\alpha$ of closed, invariant subsets ($j_\alpha : \Omega_\alpha \to \Omega$ the inclusion), then $\mathrm{Prim}(\mathcal{C}_0(\Omega) \rtimes G)$ is the disjoint union*

$$\mathrm{Prim}(\mathcal{C}_0(\Omega) \rtimes G) \ = \ \cup_{\alpha \in I} (j_\alpha \rtimes G)^* (\mathrm{Prim}(\mathcal{C}_0(\Omega_\alpha) \rtimes G)) .$$

Proof. This follows from Proposition 6.1 using the fact that $\Omega \times \hat{G} = \cup_{\alpha \in I} \Omega_\alpha \times \hat{G}$. □

Let ϕ be a $*$-morphism between two C^*-algebras A and B. The *support* $\mathrm{supp}(\phi) = \mathrm{Prim}_{\ker(\phi)}(A)$ of ϕ is the set of primitive ideals containing $\ker(\phi)$. If ϕ is surjective, its support is the image of $\phi^* : \mathrm{Prim}(B) \to \mathrm{Prim}(A)$ (which is defined in this particular case). We shall need the concept of exhausting families [23].

Definition 6.3. *A set of $*$-morphisms $\mathcal{F} = \{\phi : A \to B_\phi\}$ of a C^*-algebra A is called* exhausting *if $\mathrm{Prim}(A) = \cup_{\phi \in \mathcal{F}} \mathrm{supp}(\phi)$.*

We thus obtain the following corollary.

Corollary 6.4. *Let us use the notation of Corollary 6.2. Then the family of morphisms $\{j_\alpha \rtimes G |\ \alpha \in I\}$ is exhausting.*

Proof. The support of $j_\alpha \rtimes G$ is the image of $\mathrm{Prim}(\mathcal{C}_0(\Omega_\alpha) \rtimes G)$. The result then follows from Corollary 6.2. □

Let $\Omega := \widehat{\mathcal{E}_{\mathcal{S}}(X)} \smallsetminus X$. We now proceed to study $\mathrm{Prim}(\mathcal{E}_{\mathcal{S}}(X) \rtimes X)$ and $\mathrm{Prim}(\mathcal{C}(\Omega) \rtimes X)$ using the results in [10, 35]. We first establish that the orbits are locally closed, using the results of the previous sections.

First of all, recall that, by Theorem 3.9, the set $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ identifies with the set of augmented $\mathcal{S}$-chains $(a, \overrightarrow{\alpha})$, with X acting only on $a \in X/Z(\overrightarrow{\alpha})$ by translations, see (19). Hence the set of all orbits of X acting on $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ is in bijection with the set of all $\mathcal{S}$-chains. In particular, each of the sets $\mathcal{X}_{\mathcal{Z}}$ introduced in Equation (24) is X invariant and has *closed* orbits. Here, of course, $\mathcal{Z}$ is the $\mathcal{S}$-flag associated to any augmented $\mathcal{S}$-chain in an orbit of $\mathcal{X}_{\mathcal{Z}}$. Corollary 4.6 then yields the following result.

Lemma 6.5. *The orbits of X acting on $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ and $\Omega := \widehat{\mathcal{E}_{\mathcal{S}}(X)} \smallsetminus X$ are locally closed.*

Recalling that the set of all orbits of X acting on $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ is in natural bijection with the set of all $\mathcal{S}$-chains, Equation (19), gives that the stabilizer of the orbit associated to the $\mathcal{S}$-chain $\overrightarrow{\alpha}$ is $Z(\overrightarrow{\alpha})$. Therefore $\mathrm{Prim}(\mathcal{E}_{\mathcal{S}}(X) \rtimes X)$ identifies the set of pairs $(\overrightarrow{\alpha}, \zeta)$, where $\overrightarrow{\alpha}$ is an $\mathcal{S}$-chain and ζ a character of $Z(\overrightarrow{\alpha})$.

In particular, since X is a single orbit in $\widehat{\mathcal{E}_{\mathcal{S}}(X)}$ (corresponding to the empty chain) and has stabilizer 0, it will contribute a single point to $\mathrm{Prim}(\mathcal{E}_{\mathcal{S}}(X) \rtimes X)$, by Proposition 6.1 applied to the open inclusion $X \subset \widehat{\mathcal{E}_{\mathcal{S}}(X)}$. Moreover, this point is the spectrum of the algebra $\mathcal{C}_0(X) \rtimes X \cong \mathcal{K}(X)$. The orbits of X on $\Omega := \widehat{\mathcal{E}_{\mathcal{S}}(X)} \smallsetminus X$ will thus correspond to the non-empty $\mathcal{S}$-chains and we have a canonical isomorphism

$$\mathcal{E}_{\mathcal{S}}(X) \rtimes X/\mathcal{K}(X) \simeq \mathcal{C}(\Omega) \rtimes X. \tag{30}$$

(This isomorphism is also simply a consequence of the exact sequence obtained by taking the crossed product by X of the exact sequence $0 \to \mathcal{C}_0(X) \to \mathcal{E}_{\mathcal{S}}(X) \to \mathcal{C}(\Omega) \to 0$.) See also Remarks 2.3 and 4.2.

We now study $\mathrm{Prim}(\mathcal{E}_{\mathcal{S}}(X) \rtimes X)$, which is our primary interest. Let $\alpha \in \mathbb{S}_X$ and let $\Omega_\alpha := \{\chi \in \widehat{\mathcal{E}_{\mathcal{S}}(X)} |\ \chi|_{\mathcal{C}(\overline{X})} = \chi_\alpha\}$ be the set of characters of $\mathcal{E}_{\mathcal{S}}(X)$ that restrict to χ_α on $\mathcal{C}(\overline{X})$, as in Lemma 3.4. We obtain that

$$\Omega \ = \ \cup_{\alpha \in \mathbb{S}_X} \Omega_\alpha \ , \tag{31}$$

a disjoint union of closed subsets, with Ω_α the image of τ_α^* acting on the primitive ideal spectrum of $\mathcal{E}_{\mathcal{S}/\alpha}(X/Z(\alpha))$, see Lemma 3.5.

Theorem 6.6. *Let $\mathcal{S}$ be a finite semi-lattice of sub-spaces of X such that $0, X \in \mathcal{S}$. For each $\alpha \in \mathbb{S}_X$, we consider the map $\tau_\alpha \rtimes X : \mathcal{E}_{\mathcal{S}}(X) \rtimes X \to \mathcal{E}_{\mathcal{S}_\alpha}(X) \rtimes X$. Then the family $\{\tau_\alpha \rtimes X\}_{\alpha \in \mathbb{S}_X}$ is an exhausting family of morphisms of the C^*-algebra $\mathcal{E}_{\mathcal{S}}(X) \rtimes X/\mathcal{K}(X)$.*

Proof. We have $\mathcal{E}_{\mathcal{S}}(X) \rtimes X/\mathcal{K}(X) \cong \mathcal{C}(\Omega) \rtimes X$, see Equation 30. The morphisms $\tau_\alpha \rtimes X$ then correspond to $j_\alpha \rtimes X$, where j_α is the restriction morphism from $\mathcal{C}(\Omega)$ to $\mathcal{C}(\Omega_\alpha)$. The result then follows from Corollary 6.4. (Note that we are in position to use this corollary in view of Lemma 6.5.) □

Let us notice that the morphisms $\tau_\alpha \rtimes X$ considered in the previous theorem were denoted simply τ_α before. Reverting to the original notation, for simplicity, we obtain

$$\sigma_{\mathrm{ess}}(P) \ := \ \sigma_{\mathcal{E}_{\mathcal{S}}(X) \rtimes X/\mathcal{K}(X)}(P) \ = \ \cup_\alpha \sigma(\tau_\alpha(P)) \ ,$$

for $P \in \mathcal{E}_{\mathcal{S}}(X) \rtimes X/\mathcal{K}(X)$, where the first equality is valid by our definition of the essential spectrum and the second one is valid since the family τ_α is an exhausting family of representations of $\mathcal{E}_{\mathcal{S}}(X) \rtimes X/\mathcal{K}(X)$, which allows us to use the results of [23]. It was proved in [13] (see also [23]) that we can extend this property to affiliated operators. Therefore, by taking $P := (H + \mathrm{i})^{-1}$, for H as in Theorem 1.1, we obtain

$$\sigma_{\mathrm{ess}}(H) \ = \ \cup_\alpha \sigma(\tau_\alpha(H)) \ .$$

This completes the proof of Theorem 1.1.

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