EXHAUSTIVE FAMILIES OF REPRESENTATIONS OF C^*-ALGEBRAS ASSOCIATED TO N-BODY HAMILTONIANS WITH ASYMPTOTICALLY HOMOGENEOUS INTERACTIONS

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ABSTRACT. We continue the analysis of algebras introduced by Georgescu, Nistor and their coauthors, in order to study N-body type Hamiltonians with interactions. More precisely, let $Y \subset X$ be a linear subspace of a finite dimensional Euclidean space X, and v_Y be a continuous function on X/Y that has uniform homogeneous radial limits at infinity. We consider, in this paper, Hamiltonians of the form $H = -\Delta + \sum_{Y \in \mathcal{S}} v_Y$, where the subspaces $Y \subset X$ belong to some given family $\mathcal{S}$ of subspaces. Georgescu and Nistor have considered the case when $\mathcal{S}$ consists of *all* subspaces $Y \subset X$, and Nistor and the authors considered the case when $\mathcal{S}$ is a finite semi lattice and Georgescu generalized these results to any families. In this paper, we develop new techniques to prove their results on the spectral theory of the Hamiltonian to the case where $\mathcal{S}$ is any family of subspaces also, and extend those results to other operators affiliated to a larger algebra of pseudodifferential operators associated to the action of X introduced by Connes. In addition, we exhibit Fredholm conditions for such elliptic operators. We also note that the algebras we consider answer a question of Melrose and Singer.

An new approach in the study of Hamiltonians of N-body type with interactions that are asymptotically homogeneous at infinity on a finite dimensional Euclidean space X was initiated by Georgescu and Nistor [3, 7, 4].

For any finite real vector space Z, we let $\overline{Z}$ denote its spherical compactification. A function in $C(\overline{Z})$ is thus a continuous function on Z that has uniform radial limits at infinity. Let $\mathbb{S}_Z$ be the set of half-lines in Z, that is $\mathbb{S}_Z := \{\, \hat{a},\ a \in Z, a \neq 0\,\}$ where $\hat{a} := \{ra,\, r > 0\}$. We identify $\mathbb{S}_Z = \overline{Z} \smallsetminus Z$.

For any subspace $Y \subset X$, $\pi_Y : X \to X/Y$ denotes the canonical projection. Let

$$H = -\Delta + \sum_{Y \in \mathcal{S}} v_Y \ , \tag{1}$$

where $v_Y \in C(\overline{X/Y})$ is seen as a bounded continuous function on X via the projection $\pi_Y : X \to X/Y$. The sum is over *all* subspaces $Y \subset X$, $Y \in \mathcal{S}$ and is assumed to be uniformly convergent. One of the main results of [7, 10] describe the essential spectrum of H extending the celebrated HVZ theorem [14]. The goal of this paper is to explain how these results can be extended to any family of subspaces that contains $\{0\}$ and to more general operators using C^*-algebras techniques.

Let $\mathcal{S}$ be a family of subspaces of X with $0 \in \mathcal{S}$. We define the commutative sub-C^*-algebra $\mathcal{E}_{\mathcal{S}}(X)$ of the commutative C^*-algebra $C^u_b(X)$ of bounded uniformly continous functions on X by

$$\mathcal{E}_{\mathcal{S}}(X) = \langle C(\overline{X/Y})\ , \quad Y \in \mathcal{S}\ \rangle \subset C_b^u(X). \tag{2}$$

The algebras $\mathcal{E}_{\mathcal{S}}(X)$ give an answer to a question of Melrose and Singer [9].

Theorem 1. *Let n be an integer. Let $\mathcal{S}^n$ be the semi-lattice of subspaces of X^n generated by $\mathcal{S}_i^n \cup \mathcal{S}_{ij}^n$ where*

$$\mathcal{S}_i^n = \{(x_1, \dots, x_n) \in X^n\ ;\ x_i = 0\}$$
$$\mathcal{S}_{ij}^n = \{(x_1, \dots, x_n) \in X^n\ ;\ x_i = x_j\}$$

Then the spectrum $\Omega_{\mathcal{S}^n}$ of $\mathcal{E}_{\mathcal{S}^n}(X^n)$ is a compactification of X^n satisfying the following properties :

1. *$\Omega_{\mathcal{S}^1}$ is the spherical compactification $\overline{X}$,*
2. *The action of the symmetric group $\mathfrak{S}_n$ on X^n extends continuously to $\Omega_{\mathcal{S}_n}$,*
3. *The projections $p_I^{n,k} \colon X^n \to X^k$, $p_I^{n,k}(x_1, \dots, x_n) = (x_{i_1}, \dots, x_{i_k})$ extend continuously to $p_I^{n,k} \colon \Omega_{\mathcal{S}_n} \to \Omega_{\mathcal{S}_k}$,*
4. *The difference maps $\delta_{ij}(x_1, \dots, x_n) = x_i - x_j$ from X^n to X extend continuously to the compactifications.*

Actually, the spectrum $\Omega_{\mathcal{S}^n}$ have very strong connection with the space built by Vasy in [16] and generalized by Kottke in the last section of [8].
The additive group X acts by translation on $C_b^u(X)$ and the subalgebra $\mathcal{E}_{\mathcal{S}}(X)$ is invariant. So a crossed product C^*-algebra is obtained

$$\mathcal{E}_{\mathcal{S}}(X) \rtimes X\ , \tag{3}$$

which can be regarded as an algebra of operators on $L^2(X)$. Thanks to the assumption $0 \in \mathcal{S}$, the algebra $C_0(X)$ belongs $\mathcal{E}_{\mathcal{S}}(X)$. Hence $C_0(X) \rtimes X$ is contained in $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$. It follows from the definition of crossed products algebras that the C^*-algebra $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$ is generated by two kinds of operators : multiplication operators m_f associated to functions $f \in \mathcal{E}_{\mathcal{S}}(X)$, and convolution operators

$$C_\phi u(x) := \int_X \phi(y) u(x-y) dy$$

with $\phi \in C_c(X)$, a continuous compactly supported function. An immediate computation shows that $m_f c_\phi$ (resp. $c_\phi m_f$) is a kernel operator with kernel

$$K(x,y) = f(x)\phi(y-x)\,, \qquad (\text{resp. } K(x,y) = f(y)\phi(y-x)). \tag{4}$$

Proposition 2. *(i) The subalgebra $C_0(X) \rtimes X$ is the algebra $\mathcal{K}(X)$ of compact operators on $L^2(X)$.*
(ii) For $f \in C(\overline{X})$ and $\phi \in C_c(X)$ the commutator $[m_f, c_\phi]$ is compact.

The point (i) is a consequence of equation (4) because the kernel K has compact support when f does and the result follows by density. Again, thanks to equation (4), one sees that the commutator is a kernel operator with kernel

$$K(x,y) = \phi(y-x)(f(x) - f(y))\,.$$

Hence, in view of $\phi \in C_c(X)$, the support of K is contained in a band around the diagonal. The distance between the border of the band and the diagonal is bounded. Moreover, K goes to 0 at infinity because f has radial limits. So the commutator is a limit of Hilbert-Schmidt operators, and hence is compact.

Recall that a self-adjoint operator P on $L^2(X)$ is said to be affiliated to a C^*-algrebra A of bounded operators on $L^2(X)$ if for some (and hence any) function $h \in C_0(\mathbb{R})$ then $h(P)$ belongs to A. For example,, it follows from the identity

$$(H+i)^{-1} = (-\Delta+i)^{-1}\left(1+V(-\Delta+i)^{-1}\right)^{-1} ,$$

that H is affiliated to $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$. More generally, for any C^*-algebra A, a morphism $h\colon C_0(\mathbb{R}) \to A$ is called an operator affiliated to A. Following Connes [2] and Baaj [1] we introduce the algebra $\Psi^\infty(\mathcal{E}_{\mathcal{S}}(X); X)$ of pseudodifferential operators associated to the action of X on $\mathcal{E}_{\mathcal{S}}(X)$. We shall need the C^*-algebra of $\Psi\mathrm{DO}(\mathcal{E}_{\mathcal{S}}(X), X)$ given by the norm closure of $\Psi^0(\mathcal{E}_{\mathcal{S}}(X); X)$ and the exact sequence

$$0 \to \mathcal{E}_{\mathcal{S}}(X) \rtimes X \to \Psi\mathrm{DO}(\mathcal{E}_{\mathcal{S}}(X), X) \xrightarrow{\sigma_0} C(\mathbb{S}_X \times \widehat{\mathcal{E}_{\mathcal{S}}(X)}) \to 0 \,, \tag{5}$$

where σ_0 is the principal symbol map. Positive order pseudodifferential operators are examples of operators affiliated to the algebra of non positive order pseudodifferential operators $\Psi\mathrm{DO}(\mathcal{E}_{\mathcal{S}}(X), X)$.

Let $\alpha \in \mathbb{S}_X$. For each $x \in X$, we let $(T_x f)(y) = f(y-x)$ denote the translation on $L^2(X)$. For any operator P on $L^2(X)$, we let

$$\tau_\alpha(P) = \lim_{r\to+\infty} T_{ra}^* P T_{ra} \ , \quad \text{if} \ \ \alpha = \hat{a} \in \mathbb{S}_X \ , \tag{6}$$

whenever the *strong* limit exists.

Lemma 3. *For $f \in C(\overline{X/Y})$ one has*

$$\tau_\alpha(f)(x) = \begin{cases} f(x) & \text{if } Y \supset \alpha\,, \\ f(\pi_Y(\alpha)) & \text{else.} \end{cases}$$

We define $\mathcal{S}_\alpha = \{Y \in \mathcal{S}\,;\, \alpha \subset Y\}$. It follows from the previous lemma that on $\mathcal{E}_{\mathcal{S}}(X)$, τ_α is the projection on the subalgebra $\mathcal{E}_{\mathcal{S}_\alpha}(X)$,

$$\tau_\alpha\colon \mathcal{E}_{\mathcal{S}}(X) \to \mathcal{E}_{\mathcal{S}_\alpha}(X)\,.$$

Theorem 4. (1) *Let P be a self-adjoint operator affiliated to $\Psi\mathrm{DO}(\mathcal{E}_{\mathcal{S}}(X), X)$ and $\alpha = \hat{a} \in \mathbb{S}_X$. Then the limit $\tau_\alpha(P) := \lim_{r\to+\infty} T_{ra}^* P T_{ra}$ exists.*

(2) *Let $P \in \Psi\mathrm{DO}(\mathcal{E}_{\mathcal{S}}(X), X)$. Then P is a Fredholm operator if and only if P is elliptic (i.e. $\sigma_0(P)$ is invertible) and for all $\alpha \in \mathbb{S}_X$, $\tau_\alpha(P)$ is invertible.*

(3) *If $P \in \Psi\mathrm{DO}(\mathcal{E}_{\mathcal{S}}(X), X)$,*

$$\mathrm{Spec}_{\mathrm{ess}}(P) = \cup_{\alpha\in\mathbb{S}_X} \mathrm{Spec}(\tau_\alpha(P)) \cup Im(\sigma_0(P)) \ .$$

(4) *If $P \in \Psi^m(\mathcal{E}_{\mathcal{S}}(X); X)$, $m > 0$, is elliptic, then,*

$$\mathrm{Spec}_{\mathrm{ess}}(P) = \cup_{\alpha\in\mathbb{S}_X} \mathrm{Spec}(\tau_\alpha(P)) \ .$$

Note that in classical results on the N-body problem, one usually has the closure of the union in the spectral decomposition. See however [10, 4]. See also [7, 10] for related results, where operators affiliated to $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$ were considered. In [10] only finite semi-lattice $\mathcal{S}$ are considered. The closure of the union means that the family (τ_α) is a *faithful* family of morphism of $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$. The stronger result of [10] is obtained by showing that the family $(\tau_\alpha \rtimes X)_{\alpha\in\mathbb{S}_X}$ is actually an *exhaustive* family of representations of $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$, when $\mathcal{S}$ is a finite semi-lattice. In [5], pseudodifferential operators on $\mathbb{R}$ were considered (see Remark 3.23 of that paper). In the framework of admissible locally compact group, decomposition of essential spectrum involving *exhaustive* families can be found in [11] [12]. In fact, by [13, Proposition 3.12], exhaustive families are also strictly spectral families in the following sense.

Definition 5. [13, 15]

(1) *A family $(\phi_i)_{i\in I}$ of morphisms of a C^*-algebra A is said to be* exhaustive *if any primitive ideal contains at least $\ker \phi_i$ for some $i \in I$.*
(2) *A family $(\phi_i)_{i\in I}$ of morphisms of a unital C^*-algebra A is said to be* strictly spectral *if*

$$(\forall a \in A) \qquad \mathrm{Spec}(a) = \cup_{i\in I} \mathrm{Spec}(\phi_i(a))$$

Theorem 6. *Let $\mathcal{S}$ be a family of subspaces of X with $0 \in \mathcal{S}$. Then the family $(\tau_\alpha \rtimes X)_{\alpha\in\mathbb{S}_X}$ is an exhaustive family of $\mathcal{E}_{\mathcal{S}}(X) \rtimes X/\mathcal{K}(X)$.*

Let us prove this result. Let π be an irreducible representation of $\mathcal{E}_{\mathcal{S}}(X) \rtimes X/\mathcal{K}(X)$. It extends to an irreducible representation of $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$ as well as to their multipliers algebras $\mathcal{M}(\mathcal{E}_{\mathcal{S}}(X) \rtimes X/\mathcal{K}(X))$ and $\mathcal{M}(\mathcal{E}_{\mathcal{S}}(X) \rtimes X)$. By proposition 2(i), one obtains the following commutative diagram:

$$\begin{array}{ccccccc} C(\overline{X}) & \hookrightarrow & \mathcal{E}_{\mathcal{S}}(X) & \longrightarrow & \mathcal{M}(\mathcal{E}_{\mathcal{S}}(X) \rtimes X) & & \\ \downarrow p & & & & \downarrow & & \\ C(\mathbb{S}_X) & & \xrightarrow{\quad\phi\quad} & & \mathcal{M}(\mathcal{E}_{\mathcal{S}}(X) \rtimes X/\mathcal{K}(X)) & \xrightarrow{\pi} & \mathcal{B}(\mathcal{H}_\pi) \end{array} \tag{7}$$

Lemma 7. *The image $\phi(C(\mathbb{S}_X))$ is central in $\mathcal{M}(\mathcal{E}_{\mathcal{S}}(X) \rtimes X/\mathcal{K}(X))$.*

In fact it is enough to show that any $f \in C(\overline{X})$ commutes with any element of $\mathcal{E}_{\mathcal{S}}(X) \rtimes X$ modulo a compact operator. But the result is true on the generators by Proposition 2(ii), so the lemma follows by density.

By the Schur Lemma, we deduce that $\pi \circ \phi$ is a character of $C(\mathbb{S}_X)$. Hence there exists some $\alpha \in \mathbb{S}_X$ such that $\pi|_{C(\overline{X})} = \chi_\alpha I$, where χ_α is the character of $C(\overline{X})$ given by the evaluation at $\alpha \in \mathbb{S}_X$.

Proposition 8. *One has* $\ker \tau_\alpha = (\ker \chi_\alpha)\mathcal{E}_{\mathcal{S}}(X)$.

Proof. We need to show that $\mathcal{E}_{\mathcal{S}}(X)/\ker\tau_\alpha = \mathcal{E}_{\mathcal{S}_\alpha}(X)$ and $\mathcal{E}_{\mathcal{S}}(X)/(\ker\chi_\alpha)\mathcal{E}_{\mathcal{S}}(X)$ have the same characters. By definition, for any character χ of $\mathcal{E}_{\mathcal{S}_\alpha}(X)$, there exists a unique character χ' of $\mathcal{E}_{\mathcal{S}}(X)$ such that $\chi' = \chi \circ \tau_\alpha$. In view of lemma 3, this is equivalent to the following :

$$(\forall Y \in \mathcal{S}, \alpha \not\subset Y, \forall u \in C(\overline{X/Y})) \quad \chi(u) = u(\pi_Y(\alpha)). \tag{8}$$

In particular, for $Y = 0$, we see that $\chi_{|C(\overline{X})} = \chi_\alpha$. Reciprocally it follows from [7, Lemma 6.7] that if $\chi_{|C(\overline{X})} = \chi_\alpha$ then relation (8) is true. On the other hand, the characters of $\mathcal{E}_{\mathcal{S}}(X)/(\ker\chi_\alpha)\mathcal{E}_{\mathcal{S}}(X)$ are precisely the characters χ of $\mathcal{E}_{\mathcal{S}}(X)$ such that $\chi_{C(\overline{X})} = \chi_\alpha$. So $\ker\tau_\alpha = (\ker\chi_\alpha)\mathcal{E}_{\mathcal{S}}(X)$ as claimed. □

Now if $\pi_{|C(\overline{X})} = \chi_\alpha$, one has $\ker \pi \supset (\ker \chi_\alpha)\mathcal{E}_{\mathcal{S}}(X) = \ker \tau_\alpha$. Finally,

$$\ker(\tau_\alpha \rtimes X) = (\ker \tau_\alpha) \rtimes X \subset \ker \pi \,.$$

It follows that $(\tau_\alpha \rtimes X)_{\alpha \in \mathbb{S}_X}$ is an exhaustive family of morphisms.

Remark 9. *The results presented here can easily be extended to pseudodifferential operators with matrix coefficients. For example, Dirac operators $D_V = D + V$, with potentials V as in* (1) *may be considered and satisfy the condition of Theorem 4.*

See also [6, Example 6.35] for others physical interesting operators.

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