TRANSLATION OF DOLBEAULT REPRESENTATIONS

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Abstract. We adapt techniques used in the study of the cubic Dirac operator on homogeneous reductive spaces to Dolbeault operators on elliptic coadjoint orbits. We prove that cohomologically induced representations have an infinitesimal character, that cohomological induction and Zuckerman translation functor commute and we give a geometric interpretation of the Zuckerman translation functor for cohomologically induced representations .

Introduction

In their proof of a conjecture of Vogan on Dirac cohomology for semisimple Lie groups Huang and Pandžić [HP02] introduced a differential complex whose differential is given by the graded commutator with the algebraic Dirac operator. The cohomology of this complex is computed using its commutative analogue given by the symbol map. Later Alekseev and Meinrenken [AM00, AM05] gave an interpretation of their computation in terms of the non-commutative Weil algebra and the Chern-Weil homomorphism that eventually leads to an easy proof of the Duflo theorem for quadratic Lie algebras as well as a theorem of Rouviere for symmetric pairs. Here we apply these powerful techniques to Dolbeault operators instead and recover easily some already known results on representation theory. In particular we show that Dolbeault cohomology representations have an infinitesimal character, that the Zuckerman translation functor is well defined in this geometric context and commutes with Dolbeault cohomology induction. Moreover we give a simple geometric interpretation of the Zuckerman translation functor for these modules. In fact it turns out to be the projection onto the fibers of the vector bundles involved. This also includes a proof of a theorem of Casselman and Osborne as well as results of Kostant on $\mathfrak{u}$-cohomology. Similar proofs of Casselman-Osborne theorem also appear in [HPR05] and an unpublished work of M. Duflo [Duf83].

Let G be a connected real reductive Lie group with complexified Lie algebra $\mathfrak{g}$. By reductive we mean that $\mathfrak{g}$ decomposes as $[\mathfrak{g}, \mathfrak{g}] + \mathfrak{z}$, where $\mathfrak{z}$ denotes the center of $\mathfrak{g}$. If K/Z is a maximal compact subgroup of G/Z, where Z is the center of G, then K is the fixed point group of

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a Cartan involution θ of G. Write $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ for the corresponding Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{k}$ is the complexified Lie algebra of K. Assume that G is a real form of a complex reductive Lie group $G^{\mathbb{C}}$. We have $\mathrm{Lie}(G^{\mathbb{C}}) = \mathfrak{g}$. We fix an $\mathrm{Ad}(G)$-invariant bilinear form B on $\mathfrak{g}$ that coincides with the Killing form on $[\mathfrak{g}, \mathfrak{g}]$. For reductive subgroups of G their corresponding complexified Lie algebra will be denoted by the corresponding German letter.

If H is a closed connected reductive θ-stable subgroup of G and (V, τ) a finite-dimensional representation of H, we then have a finite rank complex vector bundle $\mathcal{V} = G \times_H V$ over G/H. We will consider smooth sections of such a vector bundle. This space $\Gamma(V)$ of smooth sections is identified with the spaces $(C^\infty(G) \otimes V)^H$ or $C^\infty(G,\, V)^H$. If V is a smooth infinite-dimensional representation of H, this last space is still well defined, and the tensor product in the first space stands for the projective tensor product (see [Gro95]). They are again canonically isomorphic and we continue to call them the space of sections of the associated infinite-dimensional bundle. The homogeneous spaces we will consider in the sequel are elliptic coadjoint orbits. These spaces may be realized as measurable open G-orbits $Y = G/H$ in a complex flag manifold $Z = G^{\mathbb{C}}/Q$ of the complexified Lie group $G^{\mathbb{C}}$. The definition of an open measurable G-orbit is given in [Wol69] as well as the following facts. A base point $z_0 \in Y$ may be chosen so that $Q = \mathrm{Stab}_{G^{\mathbb{C}}}(z_0)$, and $H = Q \cap \overline{Q} = \mathrm{Stab}_G(z_0)$ contains a fundamental Cartan subgroup T of G. One may also assume that $\mathfrak{h}$ is the centralizer of a compact torus $\mathfrak{t}' \subset \mathfrak{t} \cap \mathfrak{k}$. So there exists $\xi_0 \in \mathfrak{t}'$ such that $\mathrm{ad}\,\xi_0$ has real eigenvalues, $\mathfrak{h}$ is the centralizer of ξ_0 and $\mathfrak{u}$ is the sum of the eigenspaces of $\mathrm{ad}\,\xi_0$ corresponding to the positive eigenvalues, and $\mathfrak{q} = \mathfrak{h} \oplus \mathfrak{u}$. It can then be shown that the homogeneous space $Y = G/H$ is isomorphic to a coadjoint elliptic orbit and that all coadjoint elliptic orbit arise in this way.

As an open submanifold of a complex manifold, Y is a complex manifold as well with a G-invariant complex structure. Under these assumptions the antiholomorphic tangent space at the origin is $\mathfrak{u}$. Moreover $\mathfrak{u}$ and $\overline{\mathfrak{u}}$ are isotropic subspaces in duality under B. So we will identify $\mathfrak{u}^*$ and $\overline{\mathfrak{u}}$. Let us now define a twisted Dolbeault operator. It is a G-invariant differential operator on the bundle $G \times_H (\wedge^\bullet \mathfrak{u}^* \otimes E)$ where E is a smooth representation of H. Let X_i be a basis of $\mathfrak{u}$, ξ_i the dual basis. We denote by $r(X_i)$ the left invariant vector field on G defined by right derivative and $e(\xi_i)$ (resp. $\iota(X_i)$) the exterior product (resp. interior product) on $\wedge^\bullet \mathfrak{u}^*$. The twisted Dolbeault operator is defined by

$$\overline{\partial}(E) = \sum_i r(X_i) \otimes e(\xi_i) \otimes I_E - \sum_{i<j} 1 \otimes e(\xi_i) e(\xi_j) \iota([X_i, X_j]) \otimes I_E \tag{1}$$

It defines a differential complex on the bundle $G \times_H (\wedge^\bullet \mathfrak{u}^* \otimes E)$ whose cohomology $H_{\overline{\partial}(E)}$ is known as Dolbeault cohomology. Again when E is any smooth representation of H, the operator in equation (1) is well defined. Wong proved [Won95, Won99] that the cohomology of the Dolbeault operator is a smooth admissible G-module which is a maximal globalization of its underlying Harish-Chandra module namely the cohomologically induced Zuckerman module $\mathcal{R}^*(E)$. The main step is to prove that the operator $\overline{\partial}(E)$ has closed range, which is a deep and difficult result. Once this is done, the results on the cohomologically induced Zuckerman module $\mathcal{R}^*(E)$ apply. In particular, Wong deduces that when E has an infinitesimal character then the cohomology space $H_{\overline{\partial}(E)}$ also has an infinitesimal character. One may also probably deduce the compatibility of Dolbeault cohomology with the Zuckerman translation functor from Wong's result. The main goal of this paper is to recover (resp. prove) these results on infinitesimal character and translations with a method that connects the Dolbeault theory with Dirac theory.

More precisely we obtain the following results. The representation of G on $(C^\infty(G) \otimes \wedge^\bullet \mathfrak{u}^* \otimes E)^H$ commutes with the Dolbeault operator. Hence its derivative also commutes with the Dolbeault operator and goes down to a well defined representation of the enveloping algebra $\mathcal{U}(\mathfrak{g})$. Its restriction to the center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ is given by the following theorem.

Theorem 0.1. *Let E be a smooth representation of H which is $\mathcal{Z}(\mathfrak{h})$ finite. Let us write $E = \oplus_\mu E(\mu)$ for its decomposition in primary component such that $E(\mu)$ has generalized infinitesimal character χ_μ. Assume that each $E(\mu)$ is a smooth H-module. Then the G-module $H_{\overline{\partial}(E(\mu))}$ has a generalized infinitesimal character corresponding to $\mu + \rho(\mathfrak{u})$ under the Harish-Chandra isomorphism and $H_{\overline{\partial}(E)} = \oplus_\mu H_{\overline{\partial}(E(\mu))}$.*

Let $E(\mu)$ be a smooth representation of H with generalized infinitesimal character χ_μ. Let F^ν be a finite-dimensional irreducible representation of G with highest weight ν. Its restriction to the connected group H has a subrepresentation isomorphic to the representation E^ν. Hence we get a homomorphism of smooth representations of H

$$i^H_{\mu,\nu} \colon E(\mu) \otimes E^\nu \to E(\mu) \otimes F^\nu \,.$$

This map induces a G-equivariant map (see Proposition 3.6)

$$i^G_{\mu,\nu} \colon C^\infty(G, \wedge^\bullet \mathfrak{u}^* \otimes E(\mu) \otimes E^\nu)^H \to C^\infty(G, \wedge^\bullet \mathfrak{u}^* \otimes E(\mu))^H \otimes F^\nu$$

Theorem 0.2. *The map $i^G_{\mu,\nu}$ goes down to cohomology*

$$i^G_{\mu,\nu} \colon H_{\overline{\partial}(E(\mu) \otimes E^\nu)} \to H_{\overline{\partial}(E(\mu))} \otimes F^\nu \,.$$

Let $E(\mu+\nu)$ be the component of $E(\mu)\otimes E^\nu$ with generalized infinitesimal character $\chi_{\mu+\nu}$. Let $i^G_{\mu+\nu}$ be the restriction of $i^G_{\mu,\nu}$ to the subbundle of $G\times_H(\wedge^\bullet\mathfrak{u}^*\otimes E(\mu)\otimes E^\nu)$ corrsesponding to $E(\mu+\nu)$. It still goes down to a well defined map in cohomology thanks to Theorem 0.1. Moreover thanks to Theorem 0.1 again and a theorem of Kostant, the representation $H_{\overline{\partial}(E(\mu))}\otimes F^\nu$ is a $\mathcal{Z}(\mathfrak{g})$ finite $\mathcal{U}(\mathfrak{g})$-module. We look at its primary component with generalized infinitesimal character $\chi_{\mu+\nu+\rho(\mathfrak{u})}$. Assume that

$$\text{(C)} \qquad \mu+\rho(\mathfrak{u})+\nu \text{ is at least singular as } \mu+\rho(\mathfrak{u})\,.$$

Theorem 0.3. *Under condition* (C) *the map*

$$i^G_{\mu+\nu}\colon H_{\overline{\partial}(E(\mu+\nu))}\to H_{\overline{\partial}(E(\mu))}\otimes F^\nu$$

is a one-to-one G-map onto the $\mathcal{Z}(\mathfrak{g})$-primary component of $H_{\overline{\partial}(E(\mu))}\otimes F^\nu$ corresponding to $\mu+\rho(\mathfrak{u})+\nu$ under the Harish-Chandra isomorphism.

Let Ψ be the Zuckerman translation functor. Theorem 0.3 reads as follows.

Corollary 0.4. *Under condition* (C) *the map $i^G_{\mu+\nu}$ induces a G-isomorphism*

$$\Psi^{\mu+\rho(\mathfrak{u})+\nu}_{\mu+\rho(\mathfrak{u})}(H_{\overline{\partial}(E)})\simeq H_{\overline{\partial}(\Psi^{\mu+\nu}_{\mu}(E))}\,.$$

This theorem is the geometric analogue of the corresponding theorem on cohomological induction. A precise statement is given in [KV95] as Theorem 7.237. The proofs we will give here apply as well if one replaces the space of sections $C^\infty(G,\wedge^\bullet\mathfrak{u}^*\otimes E)^H$ by the space $\mathrm{Hom}_H(\mathcal{U}(\mathfrak{g})\otimes\wedge^\bullet\mathfrak{u},\,E)$. The abstract Dolbeault operator still defines on this space a differential complex whose cohomology is precisely the cohomologically induced module $\mathcal{R}^\bullet(E)$. So as we claimed in the abstract we will prove Theorem 7.237 in [KV95] as well.

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1. Differential operators on homogeneous spaces

Let $\sharp\colon\mathcal{U}(\mathfrak{h})\to\mathcal{U}(\mathfrak{h})$ be the antipode. This is the antiautomorphism of $\mathcal{U}(\mathfrak{h})$ given by $X^\sharp=-X$ on $\mathfrak{h}$. Hence for $X_1,\dots,X_n\in\mathfrak{h}$,

$$(X_1\cdots X_n)^\sharp=(-1)^nX_n\cdots X_1\,.$$

We consider $\mathcal{U}(\mathfrak{g})$ as a right $\mathcal{U}(\mathfrak{h})$-module for the right multiplication. The space $\mathrm{End}(V)$ of continuous linear endomorphisms of the smooth representation (V,τ) of the group H is viewed as a left $\mathcal{U}(\mathfrak{h})$-module with the action given by

$$(\forall h \in \mathcal{U}(\mathfrak{h}),\, T \in \mathrm{End}(V)) \qquad h \cdot T = T \circ \tau(h^{\sharp})\,.$$

Let J be the left ideal of $(\mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(V))$ generated by elements of the form

$$u \cdot h \otimes T - u \otimes h \cdot T \qquad (u \in \mathcal{U}(\mathfrak{g})\,, h \in \mathcal{U}(\mathfrak{h})\,, T \in \mathrm{End}(V))$$

Then the amalgamated tensor product over $\mathcal{U}(\mathfrak{h})$ is given by

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathrm{End}(V) = (\mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(V))\,/J$$

Let q be the quotient map from $\mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(V)$ to $(\mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(V))\,/J$. The action $\mathrm{Ad} \otimes \mathrm{Ad}$ of H on the tensor product leaves J stable, and hence induces an action of H on the quotient. The invariant space for this action is the image of the H-invariants in the tensor product under q so that

$$q\colon\ (\mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(V))^H \to \big(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathrm{End}(V)\big)^H$$

First note the following lemma.

Lemma 1.1. [KR00] *1. The left $\mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(V)$-ideal J is generated by the elements of the form*

$$Y \otimes I + 1 \otimes \tau(Y) \qquad (Y \in \mathfrak{h})\,.$$

2. The module J^H of H-invariant elements in J is a two-sided ideal of the algebra $(\mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(V))^H$ and is the kernel of q.

This implies in particular that the space $\big(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathrm{End}(V)\big)^H$ is an algebra. For $X \in \mathrm{Lie}(G)$ let $r(X)$ be the left invariant vector field on G given by right differentiation

$$r(X)f(g) = \left[\frac{d}{dt} f\left(g \exp(tX)\right)\right]_{t=0}\,.$$

Let us extend r to $\mathfrak{g}$ by linearity and to the enveloping algebra $\mathcal{U}(\mathfrak{g})$ by

$$r(X_1 X_2 \cdots X_k) = r(X_1) \circ r(X_2) \circ \cdots r(X_k), \ \ \forall X_i \in \mathfrak{g}.$$

One can also attach to X a right invariant vector field on G defined as

$$l(X)f(g) = \left[\frac{d}{dt} f\left(\exp(-tX) g\right)\right]_{t=0}\,.$$

and left and right invariant derivatives are related by

$$(l(u)f)(g) = (r(\mathrm{Ad}(g)u^{\sharp})f)(g) \text{ for all } u \in \mathcal{U}(\mathfrak{g}).$$

In particular when $Z \in \mathcal{Z}(\mathfrak{g})$ lies in the center of the enveloping algebra :

$$l(Z) = r(Z^\sharp)\,.$$

Proposition 1.2. *(see e.g.* [KR00]*) The algebra $\mathbb{D}_G(\mathcal{V})$ of G-invariant differential operators on $\mathcal{V}$ is isomorphic to the algebra $\big(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathrm{End}(V)\big)^H$. This isomorphism is induced by the following representation of $\mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(V)$ on $C^\infty(G, V)$:*

$$(X \in \mathfrak{g},\, T \in \mathrm{End}(V),\, f \in C^\infty(G, V)) \qquad (X \otimes T)f(g) = T(r(X)f(g))\,.$$

In the sequel the representation (V, τ) will be $(\wedge^\bullet \overline{\mathfrak{u}}, \wedge \mathrm{Ad}\,|_H)$ in the simplest case but in order to introduce twisted versions of the differential operators we will need an extra representation of H. Let (E, σ) be any smooth representation of H and let φ_E be the algebra homomorphism defined by

$$\varphi_E\colon\ (\mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(V))^H \xrightarrow{q \circ (r \otimes c \otimes I_E)} \big(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathrm{End}(V \otimes E)\big)^H\ .$$

Here c stands for the canonical action of $\mathrm{End}(V)$ on V. An element in $(\mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(V))^H$ is named here an abstract differential operator. If $D \in (\mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(V))^H$ we then have an invariant differential operator on the vector bundle $\mathcal{V} \otimes \mathcal{E}$, where (E, σ) is a smooth representation of H.

$$D(E) = \varphi_E(D)\,.$$

Note that when $E = E_1 \oplus E_2$ is a decomposable smooth representation of H then

$$D(E) = D(E_1) \oplus D(E_2)\,. \tag{2}$$

Another by-product of this construction is to provide algebraic operators when a smooth representation (X, π) of G is given. Actually we define D_X by

$$\begin{array}{rccc} \pi \otimes 1\colon & \mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(V) & \longrightarrow & \mathrm{End}(X \otimes V) \\ & D & \longmapsto & D_X\,. \end{array}$$

If X and E are given it is also useful to consider the operator

$$D_X \otimes I_E = \pi \otimes c \otimes I_E(D) \in \mathrm{End}(X \otimes V \otimes E)\,. \tag{3}$$

In the context of Dolbeault operators where $V = \wedge^\bullet \overline{\mathfrak{u}}$, we obtain an algebra homomorphism

$$\varphi_E\colon\ (\mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(\wedge \overline{\mathfrak{u}}))^H \xrightarrow{q \circ (r \otimes c \otimes I_E)} \big(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathrm{End}(\wedge \overline{\mathfrak{u}} \otimes E)\big)^H\ .$$

The abstract operators we shall consider are

$$\hat{\overline{\partial}} = \sum_i X_i \otimes e(\xi_i)\,, \tag{4a}$$

$$v = -\sum_{i<j} 1 \otimes e(\xi_i)e(\xi_j)\iota([X_i, X_j])\,, \tag{4b}$$

(4c) and the Dolbeault operator $\overline{\partial} = \hat{\overline{\partial}} + v\,.$

2. The infinitesimal character

First we note some basic facts of linear algebra. If x is a linear operator on a vector space such that $x^2 = 0$, we denote its cohomology by $H_x = \frac{\ker x}{\operatorname{im} x}$. Let $\mathcal{A}$ be a $\mathbb{Z}_2$-graded algebra with grading operator γ. By definition this means that $\gamma^2 = 1$, and we say that γ is 1 on even elements and -1 on odd elements. The graded commutator $[\,,\,]$ turns $\mathcal{A}$ into a Lie superalgebra. We define d_a as the graded commutator with a, that is

$$d_a(b) = [a, b]\,.$$

This means that if $a \in \mathcal{A}$ is an odd element then

$$d_a(b) = [a, b] = ab - \gamma(b)a \qquad (b \in \mathcal{A})\,.$$

If a is even then

$$d_a(b) = [a, b] = ab - ba \qquad (b \in \mathcal{A})\,.$$

In any case d_a is a graded endomorphism of the $\mathbb{Z}_2$-graded vector space $\mathcal{A}$, and has the same degree as a (the space of endomorphisms is also graded). So $a \mapsto d_a$ is an (even) homomorphism of the underlying super Lie algebras. This means for example that if a and b are odd elements of A, then

$$d_a d_b + d_b d_a = d_{[a,b]} \quad \text{so in particular} \quad d_a^2 = d_{a^2} = 0 \text{ if } a^2 = 0\,. \tag{5}$$

Proposition 2.1. *The endomorphism d_a is a graded derivation. In particular* $\ker d_a$ *is a subalgebra of* $\operatorname{End}(\mathcal{A})$*. Moreover* $\operatorname{im} d_a \cap \ker d_a$ *is a two sided ideal of* $\ker d_a$*. In particular, when $a^2 = 0$, then H_{d_a} is an algebra.*

Proof. For $b, c \in \mathcal{A}$ one has for a given odd a

$$d_a(bc) = abc - \gamma(bc)a = d_a(b)c + \gamma(b)ac - \gamma(b)\gamma(c)a = d_a(b)c + \gamma(b)d_a(c)\,.$$

Assume $b, c \in \ker d_a$. Then it immediately follows that $bc \in \ker d_a$. Now if moreover $b = d_a(b')$ for some $b' \in \ker d_a^2$, then, $bc = d_a(b')c + \gamma(b')d_a(c) = d_a(b'c)$ and, using $\gamma(c) \in \ker d_a$ (because

γ anticommutes with d_a),

$$cb = cd_a(b') = d_a(\gamma(c)b') - d_a(\gamma(c))b' = d_a(\gamma(c)b')\,.$$

An analogous computation holds when a is even. $\square$

Now assume we have a graded representation of $\mathcal{A}$,

$$\pi\colon \mathcal{A} \to \mathrm{End}(V)\,,$$

on a $\mathbb{Z}_2$-graded vector space V. Then the representation π induces by restriction a representation of the algebra $\ker d_a$

$$\pi\colon\ \ker d_a \to \ker d_{\pi(a)} \subset \mathrm{End}(\ker \pi(a))\,.$$

Moreover $\pi(\operatorname{im} d_a) \subset \operatorname{im} d_{\pi(a)}$, and

(1) The ideal $\operatorname{im} d_{\pi(a)}$ sends $\ker \pi(a)$ onto $\operatorname{im} \pi(a)$.
(2) The algebra $\ker d_{\pi(a)}$ leaves $\operatorname{im} \pi(a)$ stable.

This implies that π induces a representation

$$H_{d_a} \to H_{d_{\pi(a)}} \subset \mathrm{End}\left(H_{\pi(a)}\right)\,.$$

The main example we will consider here is as follows. The space V is the space of smooth sections of the vector bundle $\wedge\overline{\mathcal{U}} \otimes \mathcal{E}$ where (E, σ) is a smooth representation of H. Moreover $\mathcal{A}$ is the algebra of abstract differential operators, and the representation of $\mathcal{A}$ will be φ_E. Note that the $\mathbb{Z}$-graduation of the tensor algebra induces a $\mathbb{Z}_2$-graduation of the exterior algebra.

Summarizing this discussion, we have obtained that φ_E induces a well defined map in cohomology

$$\overline{\varphi}_E\colon H_{d_{\overline{\partial}}} \to H_{d_{\overline{\partial}(E)}} \subset \mathrm{End}\left(H_{\overline{\partial}(E)}\right)\,.$$

The algebra $H_{d_{\overline{\partial}}}$ then acts on $H_{\overline{\partial}(E)}$. The left representation of $\mathcal{U}(\mathfrak{g})$ on $H_{\overline{\partial}(E)}$ restricts to $\mathcal{Z}(\mathfrak{g})$. If some $Z \in \mathcal{Z}(\mathfrak{g})$ is seen as an invariant differential operator acting by $r(Z) \otimes I_{\wedge^\bullet \overline{\mathfrak{u}} \otimes E}$, remember that

$$l(Z) = r(Z^\sharp) \otimes I_{\wedge^\bullet \overline{\mathfrak{u}} \otimes E}\,.$$

As any element in $\mathcal{Z}(\mathfrak{g}) \otimes I$ commutes with abstract odd differential operators and has even degree, it lies in the kernel of $d_{\overline{\partial}}$. We have proved the following

Proposition 2.2. *The restriction to $\mathcal{Z}(\mathfrak{g})$ of the (left) action of $\mathcal{U}(\mathfrak{g})$ on $\ker \overline{\partial}(E)$ goes do to a well defined action l on $H_{\overline{\partial}(E)}$. Moreover if $Z \in \mathcal{Z}(\mathfrak{g})$, then the action of $l(Z)$ on $H_{\overline{\partial}(E)}$*

only depends on the class of $Z^\sharp$ in the cohomology space $H_{d_{\overline{\partial}}}$. More precisely, this action is given by

$$l(Z) = r(Z^\sharp) \otimes I_{\wedge\overline{\mathfrak{u}}\otimes E}\,.$$

We now need to compute this representation of $\mathcal{Z}(\mathfrak{g})$ in terms of the Harish-Chandra isomorphism. If the operator $\overline{\partial}$ is replaced by the cubic Dirac operator D, the representation $\mathcal{Z}(\mathfrak{g}) \to H_{d_D}$ has been computed in the proof of the Vogan conjecture as given by Huang and Pandžić [HP06] [HP02]. In their proof they determine a homomorphism

$$\zeta_D \colon \mathcal{Z}(\mathfrak{g}) \to \mathcal{Z}(\mathfrak{h})$$

in the case $H = K$ and Kostant [Kos03] extends it to the general case. Let

$$\delta_{\mathfrak{h}} : \mathcal{U}(\mathfrak{h}) \to \mathcal{U}(\mathfrak{g}) \otimes \operatorname{End}(\wedge^\bullet \overline{\mathfrak{u}})$$

be the derivative of the (restriction of the) representation $\operatorname{Ad}\otimes\wedge\operatorname{Ad}$ of H on $\mathcal{U}(\mathfrak{g})\otimes\operatorname{End}(\wedge^\bullet\overline{\mathfrak{u}})$. (In the case of the Dirac operator this representation becomes $\operatorname{Ad}\otimes\mathrm{cliff}$ where cliff is the Clifford multiplication). The key point in the proof of Huang and Pandžić is to get a Hodge decomposition

$$\ker d_D \simeq \delta_{\mathfrak{h}}(\mathcal{Z}(\mathfrak{h})) \oplus \operatorname{im} d_D\,. \tag{6}$$

So $H_{d_D} \simeq \delta_{\mathfrak{h}}(\mathcal{Z}(\mathfrak{h})) \simeq \mathcal{Z}(\mathfrak{h})$, and the map ζ_D is defined as the composition $\mathcal{Z}(\mathfrak{g}) \to H_{d_D} \simeq \mathcal{Z}(\mathfrak{h})$. In the Subsection 2.1 we will prove the following theorem which states the analogous result for the Dolbeault operator.

Theorem 2.3. *Let $D = \overline{\partial}$ be the abstract Dolbeault operator defined in* (4)*. Then the decomposition in equation (6) is still true.*

In particular we again have $H_{d_{\overline{\partial}}} \simeq \delta_{\mathfrak{h}}(\mathcal{Z}(\mathfrak{h})) \simeq \mathcal{Z}(\mathfrak{h})$ and we can still define a map

$$\zeta_{\overline{\partial}} \colon \mathcal{Z}(\mathfrak{g}) \to \mathcal{Z}(\mathfrak{h})\,.$$

Lemma 2.4. *The map $\zeta_{\overline{\partial}}$ satisfies*

$$\zeta_{\overline{\partial}}(Z)^\sharp = \zeta_{\overline{\partial}}(Z^\sharp)\,, \qquad (Z \in \mathcal{Z}(\mathfrak{g}))\,.$$

This lemma will be clear from the identification of the map $\zeta_{\overline{\partial}}$ given in the proof of theorem 2.3 below. Let (E,σ) be any smooth representation of H as before. The action of $l(Z)$ of $Z \in \mathcal{Z}(\mathfrak{g})$ on $H_{\overline{\partial}(E)}$ is then given by

$$l(Z) = r(Z^\sharp) \otimes I_{\wedge\overline{\mathfrak{u}}\otimes E} = (r\otimes c)(\zeta_{\overline{\partial}}(Z^\sharp)) \otimes I_E = 1 \otimes I_{\wedge\overline{\mathfrak{u}}} \otimes \sigma(\zeta_{\overline{\partial}}(Z))\,.$$

Theorem 2.5 (theorem 0.1)**.** *Let E be a smooth representation of H which is $\mathcal{Z}(\mathfrak{h})$-finite. If $E = \oplus_\mu E(\mu)$ is a decomposition of E into primary $\mathcal{Z}(\mathfrak{h})$-modules with respective generalized infinitesimal characters χ_μ, then as a $(\mathcal{Z}(\mathfrak{g}), G)$-module*

$$H_{\overline{\partial}(E)} = \oplus_\mu H_{\overline{\partial}(E(\mu))} \, .$$

and the representation of $\mathcal{Z}(\mathfrak{g})$ on $H_{\overline{\partial}(E(\mu))}$ is given by the generalized infinitesimal character $\chi_{\mu+\rho(\mathfrak{u})}$.

Proof. In the case of the Dirac operator, the morphism ζ_D fits into the following commutative diagram

$$\begin{array}{ccc} \mathcal{Z}(\mathfrak{g}) & \xrightarrow{\zeta_D} & \mathcal{Z}(\mathfrak{h}) \\ \downarrow & & \downarrow \\ S(\mathfrak{t}_\mathfrak{g})^{W_G} & \longrightarrow & S(\mathfrak{t}_\mathfrak{h})^{W_H} \end{array} \tag{7}$$

Here the algebra $\mathfrak{t}_\mathfrak{g}$ (resp. $\mathfrak{t}_\mathfrak{h}$) a is Cartan subalgebra of $\mathfrak{g}$ (resp. $\mathfrak{h}$ contained in $\mathfrak{t}_\mathfrak{g}$), the vertical arrows are the Harish-Chandra isomorphisms and the bottom map is restriction. For the Dolbeault operator one needs to know what happens to this diagram. This is exactly where the $\rho(\mathfrak{u})$-shift appears. The argument is given in the proof of Theorem 2.3 in Subsection 2.1. □

2.1. **Proof of theorem 2.3.** The strategy of the proof follows closely that of Huang and Pandžić for the Dirac operator [HP02, HP06]. However in the situation we consider here the operator $d_{\overline{\partial}}$ defines a differential complex on the whole space $\mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(\wedge^\bullet \overline{\mathfrak{u}})$, not only on its H-invariant part. We will write $d_{\overline{\partial}_{\text{full}}}$ for this extended complex. So $d_{\overline{\partial}_{\text{full}}}$ is seen as an endomorphism of the algebra $\mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(\wedge^\bullet \overline{u})$. Note that the inclusion of the H-invariants in the whole complex induces a map $H_{d_{\overline{\partial}}} \to H_{d_{\overline{\partial}_{\text{full}}}}$.

The algebra $\mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(\wedge^\bullet \overline{u})$ has a natural filtration induced by the filtration of $\mathcal{U}(\mathfrak{g})$ and the trivial filtration of $\mathrm{End}(\wedge^\bullet \overline{u})$. As we shall recall in the proof of lemma 2.6 below, we have an algebra isomorphism $\mathrm{End}(\wedge^\bullet \overline{u}) \simeq \wedge^\bullet \mathfrak{h}^\perp$. So the graded algebra associated to this filtration is isomorphic to $S(\mathfrak{h}) \otimes S(\mathfrak{h}^\perp) \otimes \wedge^\bullet \mathfrak{h}^\perp$. The differential $d_{\overline{\partial}_{\text{full}}}$ preserves this filtration (shifting the degree by 1) so it induces a differential $\mathrm{Gr}(d_{\overline{\partial}_{\text{full}}})$ on the associated graded space by setting $\mathrm{Gr}(d_{\overline{\partial}_{\text{full}}})(\mathrm{Gr}\, a) = \mathrm{Gr}(d_{\overline{\partial}_{\text{full}}}(a))$. The next lemma identifies this differential. This computation has been carried out by Huang, Pandžić and Renard in [HPR06], Remark 2.3. Full details can be found in [HPR05, section 3]. For the convenience of the reader, we recall here the steps we need from this computation.

Let $\partial_{\mathfrak{u}} \in \mathrm{End}(S(\mathfrak{h}^\perp) \otimes \wedge^\bullet \mathfrak{h}^\perp)$ be the Koszul differential along $\mathfrak{u}$:

$$\partial_{\mathfrak{u}}(u \otimes \omega) = \sum_i X_i u \otimes \iota(X_i)\omega\,.$$

Lemma 2.6. *We have the following commutative diagram*

$$\begin{array}{ccc} \mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(\wedge^\bullet \overline{\mathfrak{u}}) & \xrightarrow{\mathrm{Gr}} & \mathcal{S}(\mathfrak{h}) \otimes \mathcal{S}(\mathfrak{h}^\perp) \otimes \wedge^\bullet \mathfrak{h}^\perp \\ \Big\downarrow d_{\overline{\partial}_{\mathrm{full}}} & & \Big\downarrow 1 \otimes \partial_{\mathfrak{u}} \\ \mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(\wedge^\bullet \overline{\mathfrak{u}}) & \xrightarrow{\mathrm{Gr}} & \mathcal{S}(\mathfrak{h}) \otimes \mathcal{S}(\mathfrak{h}^\perp) \otimes \wedge^\bullet \mathfrak{h}^\perp \end{array}$$

Proof. The algebra $\mathrm{End}(\wedge^\bullet \overline{\mathfrak{u}})$ is the algebra generated by the creation operators $e(\xi_i)$ and annihilation operators $\iota(X_i)$. These operators satisfy the relations :

$$(8) \qquad \begin{gathered} e(\xi_i)e(\xi_j) + e(\xi_j)e(\xi_i) = 0\,, \qquad \iota(X_i)\iota(X_j) + \iota(X_j)\iota(X_i) = 0 \\ e(\xi_i)\iota(X_j) + \iota(X_j)e(\xi_i) = \delta_{ij}\,. \end{gathered}$$

So any element of $\mathrm{End}(\wedge \overline{u})$ is in a unique way a sum of elements of the form

$$w_{IJ} = e(\xi_{i_1}) \circ \cdots e(\xi_{i_k}) \circ \iota(X_{j_1}) \circ \cdots \iota(X_{j_l}) \quad (i_1 < \cdots < i_k\,, j_1 < \cdots < j_l)\,.$$

The identification of $\mathrm{End}(\wedge \overline{\mathfrak{u}})$ with $\wedge \mathfrak{h}^\perp$ sends an element of this form to

$$s(w_{IJ}) = \xi_{i_1} \wedge \cdots \wedge \xi_{i_k} \wedge X_{j_1} \wedge \cdots \wedge X_{j_l} \in \wedge \overline{\mathfrak{u}} \otimes \wedge \mathfrak{u} \simeq \wedge \mathfrak{h}^\perp\,.$$

So for $u \in \mathcal{U}(\mathfrak{g})$, one gets

$$\mathrm{Gr}(d_{\overline{\partial}})(\mathrm{Gr}(u \otimes w_{IJ})) = \mathrm{Gr}(d_{\hat{\overline{\partial}}})(\mathrm{Gr}(u \otimes w_{IJ})) = \sum_i X_i \mathrm{Gr}(u) \otimes s([e(\xi_i), w_{IJ}])$$

where $[e(\xi_i), w_{IJ}]$ is the graded commutator, and thanks to relation (8)

$$s([e(\xi_i), w_{IJ}]) = \iota(X_i)s(w_{IJ})$$

□

Lemma 2.7. *The inclusion of $S(\mathfrak{h}) \otimes S(\overline{\mathfrak{u}}) \otimes \wedge^\bullet \overline{\mathfrak{u}}$ in $S(\mathfrak{h}) \otimes S(\mathfrak{h}^\perp) \otimes \wedge^\bullet \mathfrak{h}^\perp$ induces an isomorphism $S(\mathfrak{h}) \otimes S(\overline{\mathfrak{u}}) \otimes \wedge^\bullet \overline{\mathfrak{u}} \simeq H_{1 \otimes \partial_u}$. The inclusion of $S(\mathfrak{h})^H \otimes 1 \otimes 1$ in $\big(S(\mathfrak{h}) \otimes S(\mathfrak{h}^\perp) \otimes \wedge^\bullet \mathfrak{h}^\perp\big)^H$ induces an isomorphism in cohomology.*

This lemma is proved as Lemma 3.5 in [HPR05]. The first part of this lemma is a well known fact on Koszul cohomology. The second part actually follows from the fact that $\mathfrak{h}$ is the centralizer of an element ξ_0 (defined in the introduction) such that $\mathrm{ad}\,\xi_0$ has positive eigenvalues on u and negative eigenvalues on $\overline{\mathfrak{u}}$.

Now any element of $S(\mathfrak{h})^H \otimes 1 \otimes 1$ is the image of an element in $\delta_{\mathfrak{h}}(\mathcal{Z}(\mathfrak{h}))$. This is true because for $u \in \mathcal{U}(\mathfrak{h})$ one has $\mathrm{Gr}(\delta_{\mathfrak{h}}(u)) = \mathrm{Gr}(u \otimes 1 \otimes 1)$. Note that $\delta_{\mathfrak{h}}(\mathcal{Z}(\mathfrak{h}))$ is contained in

$\ker d_{\overline{\partial}}$. Moreover $\delta_{\mathfrak{h}}(\mathcal{Z}(\mathfrak{h})) \cap \mathrm{im} d_{\overline{\partial}} = 0$ because it is true on the right side of the diagram by Lemma 2.7.

Let $a \in \ker d_{\overline{\partial}}$. We want to show that there exists $u \in \mathcal{Z}(\mathfrak{h})$ and $b \in \mathrm{im} d_{\overline{\partial}}$ such that $a = \delta_{\mathfrak{h}}(u) + b$. We proceed by induction. This is obvious if a has order 0. Assume this is true for any operator of order less than $p-1$, and let $a \in \mathcal{U}_p \otimes \mathrm{End}(\wedge^{\bullet}\overline{\mathfrak{u}})$ such that $\mathrm{Gr}(a)$ has non vanishing cohomology class. Then $\mathrm{Gr}(a) = s + 1 \otimes \partial_{\mathfrak{u}}(\mathrm{Gr}(b))$ for some $s \in S(\mathfrak{h})^H \otimes 1 \otimes 1$ and $b \in \mathcal{U}_{p-1}(\mathfrak{g}) \otimes \mathrm{End}(\wedge^{\bullet}\overline{\mathfrak{u}})$. Let $u \in \mathcal{Z}(\mathfrak{h})$ such that $\mathrm{Gr}(\delta_{\mathfrak{h}}(u)) = s$. Then

$$d_{\overline{\partial}}\left(a - \delta_{\mathfrak{h}}(u) - d_{\overline{\partial}}(b)\right) = 0\,.$$

Moreover $a - \delta_{\mathfrak{h}}(u) - d_{\overline{\partial}}(b)$ has degree $p-1$. By assumption, there exist $u' \in \mathcal{Z}(\mathfrak{h})$ and b' such that

$$a - \delta_{\mathfrak{h}}(u) - d_{\overline{\partial}}(b) = \delta_{\mathfrak{h}}(u') + d_{\overline{\partial}}(b')$$

Hence

$$a - \delta_{\mathfrak{h}}(u+u') = d_{\overline{\partial}}(b+b')$$

So we have proved Theorem 2.3.

Now we identify the map $\zeta_{\overline{\partial}}$. Recall that $\mathcal{U}(\mathfrak{g})$ has the following decomposition [KV95, Lemma 4.123] :

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) \oplus \left(\overline{\mathfrak{u}}\,\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{u}\right). \tag{9}$$

Moreover if p is the projection onto the first component then for $z \in \mathcal{Z}(\mathfrak{g})$ we have $p(z) \in \mathcal{Z}(\mathfrak{h})$ and $z - p(z) \in \mathcal{U}(\mathfrak{g})\mathfrak{u}$. It follows from the first part of lemma 2.7 that

$$\mathrm{Gr}(z \otimes 1 \otimes 1 - \delta_{\mathfrak{h}}(p(z)) = \mathrm{Gr}((z - p(z)) \otimes 1 \otimes 1) = 0 \in H_{1\otimes\partial_{\overline{\mathfrak{u}}}}\,.$$

It follows that if $z \in \mathcal{Z}(\mathfrak{g})$ then $z = \delta_{\mathfrak{h}}(p(z)))$ in $H_{d_{\overline{\partial}}}$. We have proved

Lemma 2.8. *The map $\zeta_{\overline{\partial}}$ is determined as follows.*

$$(\forall z \in \mathcal{Z}(\mathfrak{h})) \quad \zeta_{\overline{\partial}}(z) = p(z)\,. \tag{10}$$

Remark 2.9 (Casselman-Osborne Theorem)**.** *If X is a $\mathcal{U}(\mathfrak{g})$-module then the algebra $H_{d_{\overline{\partial}}}$ acts on $H_{\overline{\partial}_{X,full}} = H(\mathfrak{u}, X)$ and for $z \in \mathcal{Z}(\mathfrak{g})$ the elements z and $\delta_{\mathfrak{h}}(p(z))$ act by the same scalar.*

Let us consider the degree 0 of $H(\mathfrak{u}, X)$ when $X = F^{\lambda}$ is an irreducible finite-dimensional G-module with highest weight $\lambda = \mu - \rho(\mathfrak{g})$. We have $H^0(\mathfrak{u}, X) = E^{\lambda}$ and this implies that for $z \in \mathcal{Z}(\mathfrak{h})$, $\delta_{\mathfrak{h}}(z)$ acts by the scalar $\chi_{\lambda+\rho(\mathfrak{h})} = \chi_{\mu-\rho(\mathfrak{u})}$. By a standard density argument it follows that the map $S(\mathfrak{t})^{W_G} \to S(\mathfrak{t})^{W_H}$ defined by prescribing the diagram (7) to be commutative is given by $\mu \mapsto \tilde{\mu} = \mu + \rho(\mathfrak{u})$. So we have identified the map $\zeta_{\overline{\partial}}$ in terms of the Harish-Chandra isomorphism. We have proved Theorem 2.5.

Remark 2.10. *This also gives a proof of the following weak version of a theorem of Kostant which computes $H_{\overline{\partial}_{F^\lambda,full}}$. Let $W_1 = \{w \in W_G\,,\ (\forall \alpha \in \Delta^+(\mathfrak{g},\mathfrak{t}))\quad w^{-1}\alpha \in \Delta^+(\mathfrak{h},\mathfrak{t})\}$. If E^ν is an irreducible H-module with highest weight ν such that $\mathrm{Hom}_H\left(E^\nu, H_{\overline{\partial}_{F^\lambda,full}}\right) \neq 0$ then there exists $w \in W_1$ such that*

$$\nu = w(\lambda + \rho(\mathfrak{g})) - \rho(\mathfrak{g})\,.$$

3. The Zuckerman translation functor

3.1. **Tensoring with finite-dimensional representations.** Let (F,π) (resp. (V,τ)) be a finite dimensional (resp. smooth) representation of G (resp. H).

Proposition 3.1. *The map $\alpha\colon C^\infty(G,\,V)^H \otimes F \to C^\infty(G,\,V\otimes F)^H$ given by*

$$\alpha(f \otimes w)(g) = f(g) \otimes \pi(g)^{-1} w \in V \otimes F$$

is a smooth G-module isomorphism.

Proof. The map α is clearly equivariant and the inverse β is given as follows. Let (w_i) be a basis of F. For any $g \in G$, the family $(\pi(g)w_i)$ is still a basis of G. If $f \in C^\infty(G,\,V\otimes F)^H$, then there exist functions f_i from G to V such that for all $g \in G$, $f(g) = \sum_i \pi(g)^{-1} w_i \otimes f_i(g)$. One checks easily that the functions f_i are H-invariant. We then set $\beta(f) = \sum f_i \otimes w_i$. The map β is well defined, equivariant and does not depend on the basis w_i. □

We now relate the different differential operators on the bundles in consideration. Recall that the algebraic operator $D_F \otimes I_E$ has been defined in equation (3).

Proposition 3.2. *Let $D \in (\mathfrak{h}^\perp \otimes \mathrm{End}(V)^H$ be an abstract operator of order 1 with vanishing order 0 part. One has*

$$\alpha \circ (D(E) \otimes I_F) = \Big(D(E\otimes F) + 1 \otimes (D_F \otimes I_E)\Big) \circ \alpha\,.$$

Proof. Let $X \in \mathfrak{h}^\perp$, $f \in C^\infty(G,\,V\otimes E)$ We have

$$\begin{aligned}(r(X)\otimes 1)\alpha(f\otimes w)(g) &= \left[\frac{d}{dt} f(g\exp(tX)) \otimes \pi(\exp(-tX)g^{-1})w\right]_{t=0}\\ &= r(X)f(g)\otimes g^{-1}w - f(g)\otimes \pi(X)\pi(g)^{-1}w\\ &= \alpha(r(X)f\otimes w)(g) - (1\otimes(I_V\otimes I_E\otimes \pi(X)))\alpha(f\otimes w)(g) \qquad (11)\end{aligned}$$

The proposition follows. □

In our case, this formula reads

$$\alpha \circ (\overline{\partial}(E) \otimes I_F) \circ \beta = \overline{\partial}(E \otimes F) + 1 \otimes (\hat{\overline{\partial}}_F \otimes I_E) = \hat{\overline{\partial}}(E \otimes F) + 1 \otimes (\overline{\partial}_F \otimes I_E)\,.$$

We will need later an algebraic version of Proposition 3.2 that does not make use of the maps α and β in the definition of the operator $\alpha(D(E) \otimes I)\beta$. Actually, using the embedding $F \hookrightarrow C^\infty(G,\, F_{|H})^H$ (given by $w \mapsto (g \mapsto \rho_w(g) = g^{-1}w)$) one sees that $C^\infty(G,\, V \otimes E)^H \otimes F$ is a subspace of the space of sections of the fibre bundle on G/H obtained by restriction to the diagonal of $G/H \times G/H$ of the bundle $(\mathcal{V} \otimes \mathcal{E}) \boxtimes \mathcal{F}$. So one expects that the Leibniz rule used in the preceding proposition has a formulation in terms of the coproduct of $\mathcal{U}(\mathfrak{g})$.

Let Δ be the coproduct of $\mathcal{U}(\mathfrak{g})$ and (F, π) a finite-dimensional representation of G as before. We define $\Delta_F = (I \otimes \pi) \circ \Delta$. Hence

$$\Delta_F \colon \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(F)\,. \tag{12}$$

Proposition 3.3. *The linear map*

$$\Delta_F \otimes I \colon \mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(V \otimes E) \to \mathcal{U}(\mathfrak{g}) \otimes \mathrm{End}(V \otimes E \otimes F)$$

induces an algebra homomorphism

$$\Delta_F \colon \mathbb{D}_G(\mathcal{V} \otimes \mathcal{E}) \longrightarrow \mathbb{D}_G(\mathcal{V} \otimes \mathcal{E} \otimes \mathcal{F})$$

Proof. We have to prove that the map $\Delta_F \otimes 1$ sends the ideal $J_{V \otimes E}$ in the definition of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{h})} \mathrm{End}(V \otimes E)$ to the ideal $J_{V \otimes E \otimes F}$ in the definition of the corresponding quotient. By lemma 1.1 it is enough to consider the elements $Y \otimes I_{V \otimes E} + 1 \otimes Y$ for $Y \in \mathfrak{h}$. But

$$\begin{aligned}\Delta_F \otimes I(Y \otimes I_{V \otimes E} + 1 \otimes (c \otimes \sigma(Y)))&\\ &= Y \otimes I_{V \otimes E \otimes F} + 1 \otimes \pi(Y) \otimes I_{V \otimes E} + 1 \otimes (c \otimes \sigma(Y))\\ &= Y \otimes I_{V \otimes E \otimes F} + 1 \otimes (\pi \otimes c \otimes \sigma(Y))\end{aligned}$$

It follows that $\Delta_F(Y \otimes I_{V \otimes E} + 1 \otimes Y) \in J_{V \otimes E \otimes F}$. ☐

Proposition 3.4. *One has* $\Delta_F(D(E)) = \alpha(D(E) \otimes I)\beta$

Proof. Let us compute $\Delta_F \otimes I(X \otimes I_{V\otimes E\otimes F})$ on $\alpha(f \otimes w)$ for $X \in \mathfrak{h}^\perp$. We have

$$\begin{aligned}\Delta_F \otimes I(X \otimes I_{V\otimes E\otimes F})\alpha(f\otimes w)&\\ &= (\Delta_F(X)\otimes I)\alpha(f\otimes w)\\ &= (r(X)\otimes I_{V\otimes E\otimes F} + 1\otimes I_{V\otimes E}\otimes \pi(X))\alpha(f\otimes w)\\ &= -(1\otimes I_{V\otimes E}\otimes\pi(X))\alpha(f\otimes w) + \alpha((r(X)f\otimes w)\\ &\qquad + (1\otimes I_{V\otimes E}\otimes\pi(X))\alpha(f\otimes w)\\ &= \alpha(r(X)f\otimes w)\end{aligned}$$

by the computation (11). The proposition follows. □

Summarising this discussion we have obtained the following result.

Lemma 3.5. *Let D be an abstract differential operator of order 1 with vanishing order 0 part. Then for any finite-dimensional representation (F,π) of G and smooth representation E of H the operators $\Delta_F(D(E))$, $D(E\otimes F)$ and $D_F\otimes I_E$ are related by the following formula :*

$$\Delta_F(D(E)) = D(E\otimes F) + 1\otimes(D_F\otimes I_E)\,.$$

3.2. **Proof of Theorems 0.2 and 0.3.** We first prove Theorem 0.2. Barchini proposed it as exercise (b) of lecture 2 in [Bar00]. The isomorphism α satisfies

$$\Delta_F(\overline{\partial}(E))\circ\alpha = \alpha\circ(\overline{\partial}(E)\otimes I_F)\,.$$

In particular, it induces an isomorphism at the level of cohomology.

Proposition 3.6. *The linear map α induces a G-module isomorphism*

$$\alpha\colon H_{\overline{\partial}(E)}\otimes F \xrightarrow{\sim} H_{\Delta_F(\overline{\partial}(E))}\,,$$

whose inverse is induced by $\beta = \alpha^{-1}$.

Moreover, thanks to Lemma 3.5 we have a decomposition

$$\Delta_F(\overline{\partial}(E)) = \overline{\partial}(E\otimes F) + 1\otimes(\hat{\overline{\partial}}_F\otimes I_E) \tag{13}$$

As in the introduction we let F^ν be a finite-dimensional irreducible representation of G with highest weight ν and consider its restriction to H. This restriction contains a finite-dimensional irreducible representation E^ν with highest weight ν. So as an H-module, one has $F^\nu = E^\nu \oplus E'$ where ν is not a weight of E'. We have an H-map

$$i^H_{\mu,\nu}\colon E(\mu)\otimes E^\nu \to E(\mu)\otimes F^\nu\,,$$

inducing a map

$$i^G_{\mu,\nu}\colon C^\infty\big(G\,,\wedge^\bullet\overline{\mathfrak{u}}\otimes E(\mu)\otimes E^\nu\big)\to C^\infty\big(G\,,\wedge^\bullet\overline{\mathfrak{u}}\otimes E(\mu)\otimes F^\nu\big)\,.$$

Thanks to equation (2), this map induces a G-map $H_{\overline{\partial}(E(\mu)\otimes E^\nu)} \longrightarrow H_{\overline{\partial}(E((\mu)\otimes F^\nu))}$, but a priori $H_{\Delta_{F^\nu}(\overline{\partial}(E(\mu)))}$ and $H_{\overline{\partial}(E(\mu)\otimes F^\nu)}$ may be different because of the presence of $\hat{\overline{\partial}}_F$ in equation (13). What we need is a map

$$H_{\overline{\partial}(E(\mu)\otimes E^\nu)} \longrightarrow H_{\Delta_{F^\nu}(\overline{\partial}(E(\mu)))}\,.$$

Proposition 3.7. *The range of $i^G_{\mu,\nu}$ is contained in the kernel of the operator $1\otimes\hat{\overline{\partial}}_{F^\nu}\otimes I_{E^\nu}$.*

Proof. The range of $i^G_{\mu,\nu}$ is the space of sections of the bundle associated to $\wedge\overline{\mathfrak{u}}\otimes(F^\nu)^{\mathfrak{u}}$. In fact $(F^\nu)^{\mathfrak{u}} = F^\nu/\overline{\mathfrak{u}}F^\nu$ is a highest weight module of H with highest weight ν. But the operator $\hat{\overline{\partial}}_{F^\nu}$ vanishes on $\wedge\overline{\mathfrak{u}}\otimes(F^\nu)^{\mathfrak{u}}$. □

Thanks to this proposition, we see that

$$\Delta_{F^\nu}(\overline{\partial}(E(\mu)))\circ i^G_{\mu,\nu} = i^G_{\mu,\nu}\circ\overline{\partial}(E(\mu)\otimes E^\nu))\,.$$

so that $i^G_{\mu,\nu}$ goes down to a G-map on cohomology. Together with Proposition 3.6, this implies Theorem 0.2.

Let us remind the condition (C) relative to Theorem 0.3 :

$$\text{(C)}\qquad \mu+\rho(\mathfrak{u})+\nu \text{ is as singular as } \mu+\rho(\mathfrak{u})\,.$$

Assume from now on that condition (C) is fulfilled. In order to prove theorem 0.3 we introduce a filtration of F^ν by $(\mathfrak{q},H)$-modules

$$E^\nu = F_0\subset F_1\subset\cdots\subset F_n = F^\nu\,,$$

such that F_k/F_{k-1} is an irreducible H-module with trivial $\mathfrak{u}$-action. Theorem 0.3 will result from the following two lemmas.

Lemma 3.8. *Let $E(\mu+\nu)$ be the component of $E(\mu)\otimes E^\nu$ with infinitesimal character $\chi_{\mu+\nu}$. Then the inclusion $E(\mu+\nu)\subset E(\mu)\otimes E^\nu$ induces an isomorphism of $H_{\overline{\partial}(E(\mu+\nu))}$ onto the component of $H_{\overline{\partial}(E(\mu)\otimes E^\nu)}$ with generalized infinitesimal character $\chi_{\mu+\nu+\rho(\mathfrak{u})}$.*

Note that the operator $\Delta_{F^\nu}(\overline{\partial}(E(\mu))$ restricts to the bundles associated to the spaces $\wedge\overline{\mathfrak{u}}\otimes E(\mu)\otimes F_k$, for $k=0,\ldots,n$. Let us denote $\Delta_{F^\nu}(\overline{\partial}(E(\mu))|_{F_k}$ this restriction. Moreover a differential operator is induced on the bundle associated to $\wedge\overline{\mathfrak{u}}\otimes E(\mu)\otimes F_k/F_{k-1}$ for $k>0$. As

F_k/F_{k-1} is trivial as a $\mathfrak{u}$-module, one sees that this induced operator is $\overline{\partial}(E(\mu)\otimes F_k/F_{k-1})$. The long exact sequence in cohomology reads

$$H_{\Delta_{F^\nu}(\overline{\partial}(E(\mu))|_{F_{k-1}}} \xrightarrow{i_k} H_{\Delta_{F^\nu}(\overline{\partial}(E(\mu))|_{F_k}} \to H_{\overline{\partial}(E(\mu)\otimes F_k/F_{k-1})} \to H_{\Delta_{F^\nu}(\overline{\partial}(E(\mu))|_{F_{k-1}}} \tag{14}$$

Lemma 3.9. *For $k>0$, the component of $H_{\overline{\partial}(E(\mu)\otimes F_k/F_{k-1})}$ with generalized infinitesimal character $\chi_{\mu+\nu+\rho(\mathfrak{u})}$ vanishes.*

From Lemma 3.9 we deduce that the restriction of i_k to the component of $H_{\Delta_{F^\nu}(\overline{\partial}(E(\mu))|_{F_{k-1}}}$ with generalized infinitesimal character $\chi_{\mu+\nu+\rho(\mathfrak{u})}$ is an isomorphism onto the corresponding component of $H_{\Delta_{F^\nu}(\overline{\partial}(E(\mu))|_{F_k}}$. Hence the component of $H_{\Delta_{F^\nu}(\overline{\partial}(E(\mu))}$ with generalized infinitesimal character $\chi_{\mu+\nu+\rho(\mathfrak{u})}$ is the component of $H_{\Delta_{F^\nu}(\overline{\partial}(E(\mu))|_{F_0}}$ with generalized infinitesimal character $\chi_{\mu+\nu+\rho(\mathfrak{u})}$. This last one is given in Lemma 3.8. Theorem 0.3 follows.

It remains to prove the two lemmas. According to Theorem 0.1, we know that if the component of $H_{\overline{\partial}(E(\mu)\otimes F_k/F_{k-1})}$ (resp. $H_{\Delta_{F^\nu}(\overline{\partial}(E(\mu))|_{F_0}}$) with infinitesimal character $\chi_{\mu+\nu+\rho(\mathfrak{u})}$ does not vanish then there exists some weight ν' of F^ν and an element w in the Weyl group such that

$$\mu+\nu'+\rho(\mathfrak{u}) = w(\mu+\nu+\rho(\mathfrak{u}))\,.$$

By condition (C) and a well known technical result [KV95, Proposition 7.166] this implies that $\nu'=\nu$ and $w=1$. The two lemmas immediately follow.

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